

JOHNSON-SCHECHTMAN INEQUALITIES FOR NONCOMMUTATIVE MARTINGALES

YONG JIAO, FEDOR SUKOCHEV, DMITRIY ZANIN, AND DEJIAN ZHOU

ABSTRACT. In this paper we study Johnson-Schechtman inequalities for non-commutative martingales. More precisely, disjointification inequalities of non-commutative martingale difference sequences are proved in an arbitrary symmetric operator space $E(\mathcal{M})$ of a finite von Neumann algebra \mathcal{M} without making any assumption on the Boyd indices of E . We show that we can obtain Johnson-Schechtman inequalities for arbitrary martingale difference sequences and that, in contrast with the classical case of independent random variables or the noncommutative case of freely independent random variables, the inequalities are one-sided except when $E = L_2(0, 1)$. As an application, we partly resolve a problem stated by Randrianantoanina and Wu in [48]. We also show that we can obtain sharp Φ -moment analogues for Orlicz functions satisfying p -convexity and q -concavity for $1 \leq p \leq 2$, $q = 2$ and $p = 2$, $2 < q < \infty$. This is new even for the classical case. We also extend and strengthen the noncommutative Burkholder-Gundy inequalities in symmetric spaces and in the Φ -moment case.

1. INTRODUCTION

Let $(f_k)_{k=0}^n \subset L_p(0, 1)$, $2 < p < \infty$, be a sequence of independent mean zero random variables. In 1970, Rosenthal [49] proved that

$$\left\| \sum_{k=1}^n f_k \right\|_p \approx_p \left\| \bigoplus_{k=0}^n f_k \right\|_{L_p \cap L_2}, \quad 2 < p < \infty,$$

where $\bigoplus_{k=0}^n f_k := \sum_{k=1}^n f_k(\cdot - k + 1)\chi_{[k-1, k)}$ is a disjoint sum of random variables $(f_k)_{k=0}^n$ which is a Lebesgue measurable function on $(0, \infty)$. Carothers and Dilworth extended the results above to the case of Lorentz spaces in [14]. In the setting of symmetric function spaces, Johnson and Schechtman [28] established a far reaching generalisation of the Rosenthal inequality. If E is a symmetric space on $[0, 1]$ (see e.g. [35, 37]) and if $1 \leq p \leq \infty$, then the set Z_E^p consists of all measurable functions on $(0, \infty)$ for which

$$\|f\|_{Z_E^p} := \|\mu(f)\chi_{[0, 1]}\|_E + \|f\|_{L_1 + L_p} < \infty.$$

Here, $\mu(f)$ denotes a decreasing rearrangement of the function f (see [35, 37] where the symbol f^* is used and Section 2 below). Johnson and Schechtman [28] proved

2010 *Mathematics Subject Classification.* Primary: 46L52, 46L53, 47A30; Secondary: 60G42.

Key words and phrases. noncommutative martingales, Johnson-Schechtman inequalities, symmetric spaces, Φ -moment inequalities.

Yong Jiao is supported by NSFC(11471337) and Hunan Province NSF(14JJ1004).

that

$$(1.1) \quad \left\| \sum_{k=0}^n f_k \right\|_E \approx_E \left\| \bigoplus_{k=0}^n f_k \right\|_{Z_E^2},$$

for every sequence $(f_k)_{k=0}^n$ of independent mean zero random variables on $(0, 1)$ whenever $L_p \subset E$ for some $p < \infty$.

Above, and in what follows, we write $A \lesssim_E B$ if there is a constant $C_E > 0$ depending only on E such that $A \leq C_E B$; and we write $A \approx_E B$ if both $A \leq C_E B$ and $B \leq C_E A$ hold, possibly with different constants; we write $A \approx B$ if the inequalities above hold for an absolute constant C which is independent of E .

Astashkin and Sukochev introduced the concept of a Kruglov operator K and extended Johnson-Schechtman inequalities in [3] (see also [7]). Namely, it is proved that (1.1) holds if and only if the Kruglov operator K is bounded on a symmetric space E (with Fatou norm). The latter condition is far less restrictive than the assumption that $L_p \subset E$ for some $p < \infty$ (see [3, Section 7]). Recently, the operator approach of [3, 7] was extended into the realm of (noncommutative) free probability theory in [52]. By using a free Kruglov operator, a version of Johnson-Schechtman inequalities in the setting of free probability theory was obtained. We briefly recall their main results. Let \mathcal{M} be a finite von Neumann algebra equipped with a faithful normal normalised trace. Let $E(\mathcal{M})$ be a symmetric Banach operator space ([34]) equipped with a Fatou norm. Then¹

$$(1.2) \quad \left\| \sum_{k=0}^n x_k \right\|_{E(\mathcal{M})} \approx \left\| \sum_{k=0}^n x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)}$$

for every sequence (x_k) of freely independent symmetrically distributed random variables from $E(\mathcal{M})$. In the special case $E = L_p$ for $1 \leq p \leq \infty$, this result was proved by Junge, Parcet and Xu [30, Theorem A]. For $p = \infty$, it is due to Voiculescu [53].

In this paper, we replace the sequence of independent (or freely independent) random variables with a radically more general case: an arbitrary sequence of noncommutative martingale differences. The notation $Int(L_p(0, 1), L_q(0, 1))$ stands for the set of all interpolation spaces for the couple $(L_p(0, 1), L_q(0, 1))$, $0 < p < q \leq \infty$. Our first main result can be stated as follows.

Theorem 1.1. *Let E be a symmetric quasi-Banach space on $(0, 1)$, let \mathcal{M} be a finite von Neumann algebra, and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be a sequence of martingale differences.*

(i) *If $E \in Int(L_1(0, 1), L_2(0, 1))$, then*

$$(1.3) \quad \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \lesssim_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)}.$$

(ii) *If $E \in Int(L_2(0, 1), L_\infty(0, 1))$, then*

$$(1.4) \quad \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)} \lesssim_E \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})}.$$

¹Here, (e_k) denotes the standard basic sequence of ℓ_∞ . In the case of $M = L_\infty(0, 1)$, the distribution of $\sum_{k \geq 0} f_k \otimes e_k$ and that of $\bigoplus_{k \geq 0} f_k$ coincide.

Even in a very special case, when \mathcal{M} is a commutative von Neumann algebra $L_\infty(0, 1)$ and E is a symmetric Banach space, Theorem 1.1 strengthens Theorem 3.1 and Theorem 3.5 (or Corollary 3.2 and Corollary 3.6) from [6]. The different method used in this present paper allows us to remove some extra assumptions (e.g., separability, order semi-continuity of the norm and the upper Boyd index $p_E < \infty$) imposed on the symmetric space E in [6]. Moreover, we consider the harder case of quasi-Banach spaces in Theorem 1.1, which renders the duality arguments used in [6] inapplicable.

In sharp contrast with the setting (1.1) (or (1.2)) of independent (or freely independent) random variables, only one-sided Johnson-Schechtman inequalities hold for martingale differences. More precisely, a two-sided disjointification inequality for noncommutative martingale differences holds only in L_2 .

Theorem 1.2. *Let E be a separable symmetric Banach space on $(0, 1)$. If for every finite von Neumann algebra \mathcal{M} and an arbitrary sequence $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ of martingale differences we have*

$$(1.5) \quad \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \approx_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)},$$

then $E = L_2(0, 1)$.

Again, in the special case of $\mathcal{M} = L_\infty(0, 1)$, Theorem 1.2 strengthens the corresponding result of [6, Corollary 3.8]. The assumptions of Theorem 1.2 is much weaker than those in [6].

The following theorem extends [44, Theorem 2.1], and strengthens [17, Proposition 4.18] and [24, Theorem 3.1]. Again, using our methods, extra assumptions on the space E imposed in [17] (e.g. 2-convexity) are eliminated.

Theorem 1.3. *Let E be a symmetric Banach space on $(0, 1)$, let \mathcal{M} be a finite von Neumann algebra, and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be a sequence of martingale differences.*

(i) *If $E \in \text{Int}(L_p(0, 1), L_2(0, 1))$ for some $1 < p < 2$, then*

$$(1.6) \quad \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \approx_E \inf_{x_k = y_k + z_k} \left(\left\| \left(\sum_{k \geq 0} |y_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |z_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right),$$

where the infimum is taken over the sequences (y_k) and (z_k) of martingale differences.

(ii) *If $E \in \text{Int}(L_2(0, 1), L_q(0, 1))$ for some $2 < q < \infty$, then*

$$(1.7) \quad \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \approx_E \max \left\{ \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}, \left\| \left(\sum_{k \geq 0} |x_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right\}.$$

Theorem 1.3 is sharp in the following sense.

Theorem 1.4. *Let E be a symmetric Banach space on $(0, 1)$. Suppose that for every sequence $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ of martingale differences the equality (1.6) (or (1.7)) holds. It follows that $E \in \text{Int}(L_p, L_q)$ for $1 < p < q < \infty$.*

Theorem 1.1 and Theorem 1.2 (respectively, Theorem 1.3 and Theorem 1.4) are proved in Section 3 (respectively, Section 4). Section 5 is devoted to the noncommutative Burkholder inequality. In the setting of L_p -spaces, it was proved by Junge and Xu [32] and was later extended to Lorentz spaces in [23] by using the weak type $(1, 1)$

decomposition of Randrianantoanina [47]. Recently, it was established in [16] (respectively, in [48]) for a symmetric operator space $E(\mathcal{M})$ such that $E \in \text{Int}(L_p, L_q)$ for $2 < p < q < \infty$ (respectively, $E \in \text{Int}(L_p, L_q)$ for $1 < p < q < 2$). However, the Burkholder inequality for the case when $E \in \text{Int}(L_2, L_q)$ for some $2 < q < \infty$ (or $E \in \text{Int}(L_p, L_2)$ for some $1 < p < 2$) was stated in [48] as an open question. Applying Theorem 1.1 and Theorem 1.3, we partly resolve this problem. Our second main result is stated as follows; see Subsection 2.6 for the unexplained notation.

Theorem 1.5. *Let E be a symmetric Banach space on $(0, 1)$, let \mathcal{M} be a finite von Neumann algebra, and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be a sequence of martingale differences. If $E \in \text{Int}(L_2(0, 1), L_4(0, 1))$, then*

$$\begin{aligned} \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} &\approx_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)} \\ &+ \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k^*|^2) \right)^{1/2} \right\|_{E(\mathcal{M})}. \end{aligned}$$

Unfortunately, at the time of writing this paper, we still do not know how to extend this result to the case $E \in \text{Int}(L_2, L_q)$ for some $4 < q < \infty$. The partial achievement in this direction is Theorem 5.10.

Our second aim in this paper is to prove some Φ -modular analogues of Johnson-Schechtman inequalities for noncommutative martingales. Let Φ be an Orlicz function on $(0, \infty)$. The study of Φ -moment inequalities for martingales was initiated by Burkholder and Gundy in their remarkable paper [13]. Since then, most of the classical p -th moment inequalities for martingales were transferred to Φ -moment inequalities; see [12, 22]. In recent years, Φ -moment inequalities have been extended to noncommutative martingales. We refer to [9, 18] for Φ -moment versions of noncommutative Burkholder-Gundy inequalities and also refer to [10] for noncommutative maximal inequalities for convex functions. In very recent papers [25, 26], the substantial difference between (1.1) as well as (1.2) and their Φ -moment analogues has been demonstrated in the case of independent and noncommutative independent random variables. Motivated by the results above, it is natural to consider Φ -moment analogue of Theorem 1.1. The following is our third main result.

Theorem 1.6. *Let \mathcal{M} be a finite von Neumann algebra and Φ be an Orlicz function, and let $\{x_k\}_{k \geq 0} \subset L_\Phi(\mathcal{M})$ be a sequence of martingale differences.*

(i) *If Φ is 2-concave, then*

$$(1.8) \quad \tau \left(\Phi \left(\left| \sum_{k \geq 0} x_k \right| \right) \right) \lesssim \mathbb{E}(\Phi(\mu(X)\chi_{(0,1)})) + \Phi(\|X\|_{L_1+L_2}), \quad X = \sum_{k \geq 0} x_k \otimes e_k.$$

(ii) *If Φ is 2-convex and q -concave for some $2 < q < \infty$, then*

$$(1.9) \quad \mathbb{E}(\Phi(\mu(X)\chi_{(0,1)})) + \Phi(\|X\|_{L_2}) \lesssim_\Phi \tau \left(\Phi \left(\left| \sum_{k \geq 0} x_k \right| \right) \right), \quad X = \sum_{k \geq 0} x_k \otimes e_k.$$

The assumptions on the Orlicz function Φ in Theorem 1.6 (ii) are sharp (see Proposition 6.6). This allows us to make some interesting comparisons between the modular inequality (1.9) and the Orlicz norm inequality (1.4). Let Φ be a 2-convex Orlicz function which fails to be q -concave for all finite q . If $L_\Phi(0, 1)$ is the Orlicz space² associated with Φ , then $L_\Phi(0, 1) \in \text{Int}(L_2(0, 1), L_\infty(0, 1))$ (see e.g. [39]).

²see Definition in Section 2

Hence, by Theorem 1.1 (ii) the inequality

$$\left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_{L_\Phi}^2(\mathcal{M} \bar{\otimes} \ell_\infty)} \lesssim_\Phi \left\| \sum_{k \geq 0} x_k \right\|_{L_\Phi(\mathcal{M})}$$

holds for an arbitrary sequence $(x_n)_{n \geq 0} \subset L_\Phi(\mathcal{M})$ of martingale differences, while the corresponding Φ -moment inequality (1.9) fails due to Proposition 6.6. This indicates an important difference between (noncommutative and classical) Johnson-Schechtman inequalities for martingales with respect to symmetric norms and their Φ -moment analogues.

Finally, in Section 7, we obtain Φ -moment Burkholder-Gundy inequalities for a p -convex and q -concave Orlicz function Φ . Our result in this section extends the main result in [9, Theorem 5.1]. We also give some examples demonstrating that our result can be applied to a larger class of Orlicz functions than that in [9].

2. PRELIMINARIES

2.1. Singular value function. Let \mathcal{M} be a finite (or semifinite) von Neumann algebra equipped with a faithful normal finite (or semifinite) trace τ . We denote by $L_0(\mathcal{M}, \tau)$, or simply $L_0(\mathcal{M})$, the family of all τ -measurable operators [21]. Recall that $e_{(s, \infty)}(|x|)$ is the spectral projection of $x \in L_0(\mathcal{M})$ associated with the interval (s, ∞) . For $x \in L_0(\mathcal{M})$, the generalized singular number is defined by

$$\mu(t, x) = \inf\{s > 0 : \tau(e_{(s, \infty)}(|x|)) \leq t\}, \quad t > 0.$$

The function $t \mapsto \mu(t, x)$ is decreasing and right-continuous [21]. For the case that \mathcal{M} is the abelian von Neumann algebra $L_\infty(0, 1)$ with the trace given by integration with respect to the Lebesgue measure, $L_0(\mathcal{M})$ is the space of all measurable functions and $\mu(f)$ is the decreasing rearrangement of the measurable function f ; see [35, 37].

2.2. Symmetric operator spaces. A quasi-Banach function space $(E, \|\cdot\|_E)$ on the interval $(0, \alpha)$, $0 < \alpha \leq \infty$ is called symmetric if for any $g \in E$ and for any measurable function f with $\mu(f) \leq \mu(g)$, we have $f \in E$ and $\|f\|_E \leq \|g\|_E$.

For a given symmetric quasi-Banach space $(E, \|\cdot\|_E)$, we define the corresponding noncommutative space on (\mathcal{M}, τ) by setting

$$E(\mathcal{M}, \tau) := \{x \in L_0(\mathcal{M}, \tau) : \mu(x) \in E\}.$$

Endowed with the quasi-norm $\|x\|_{E(\mathcal{M})} := \|\mu(x)\|_E$, the space $E(\mathcal{M}, \tau)$ is called the noncommutative symmetric space associated with (\mathcal{M}, τ) corresponding to the function space $(E, \|\cdot\|_E)$. It is shown in [51] that the quasi-norm space $(E(\mathcal{M}), \|\cdot\|_{E(\mathcal{M})})$ is complete if $(E, \|\cdot\|_E)$ is complete. For every semifinite von Neumann algebra \mathcal{N} , the space $Z_E^2(\mathcal{N})$ is well defined due to [34].

The following useful construction can be found in [28]. If E is a symmetric quasi-Banach space on the interval $(0, 1)$, then the space Z_E^2 consists of all measurable functions $f \in L_1(0, \infty) + L_\infty(0, \infty)$ for which

$$\|f\|_{Z_E^2} := \|\mu(f)\chi_{(0,1)}\|_E + \|f\|_{L_1+L_\infty} < \infty.$$

It is obvious that the functional $\|\cdot\|_{Z_E^2}$ is a quasi-norm on Z_E^2 . For $f \in L_1(0, \infty) + L_\infty(0, \infty)$ and $t > 0$, it is well known that

$$(2.1) \quad \frac{1}{4} \left(\int_0^1 \mu(t, f) dt + \left(\int_1^\infty \mu^2(t, f) dt \right)^{1/2} \right) \\ \leq \|f\|_{L_1+L_2} \leq \int_0^1 \mu(t, f) dt + \left(\int_1^\infty \mu^2(t, f) dt \right)^{1/2}.$$

Thus it is easy to see that

$$\|f\|_{Z_E^2} \approx \|\mu(f)\chi_{(0,1)}\|_E + \|f\chi_{[1,\infty)}\|_{L_2}.$$

In what follows, our other unexplained terminologies concerning symmetric function spaces are standard; see for example [37] for definitions of Boyd indices, p -convexity, etc.

2.3. Interpolation. Let F_1, F_2 and $F_1 \cap F_2 \subset E \subset F_1 + F_2$ be symmetric quasi-Banach spaces. We say that E is an interpolation space between F_1 and F_2 (written $E \in \text{Int}(F_1, F_2)$) if for every $T : F_1 + F_2 \rightarrow F_1 + F_2$ such that $T : F_1 \rightarrow F_1$ and $T : F_2 \rightarrow F_2$ is bounded, we also have $T : E \rightarrow E$ is bounded. We refer to [35] and [33] for some necessary background on interpolation. For $1 \leq r < \infty$, let $E_{(r)}$ be r -concavification of E , that is, $\|f\|_{E_{(r)}} = \| |f|^{1/r} \|_E^r$. It is proved [15, Theorem 4.23] that

$$(2.2) \quad E \in \text{Int}(L_p, L_q) \iff E_{(r)} \in \text{Int}(L_{\frac{p}{r}}, L_{\frac{q}{r}}).$$

Similarly, let $E^{(r)}$ be r -convexification of E , that is, $\|f\|_{E^{(r)}} = \| |f|^r \|_E^{1/r}$.

2.4. Orlicz functions and Orlicz spaces. Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$ be an Orlicz function, that is, Φ is an even convex function such that $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. Given an Orlicz function Φ and an operator $x \in L_0(\mathcal{M}, \tau)$, we have by [21, Corollary 2.8],

$$\tau(\Phi(|x|)) = \int_0^\infty \Phi(\mu(t, x)) dt.$$

Obviously, if $\Phi(t) = t^p$ for $1 \leq p < \infty$ then this reduces to the usual p -moment of $|x|$. By induction, it follows from [21, Theorem 4.4 (iii)] that for every sequence $(x_i)_{i=0}^n \subset L_0(\mathcal{M}, \tau)$ and scalars $\lambda_i \in (0, 1)$ with $\sum_{i=0}^n \lambda_i \leq 1$,

$$(2.3) \quad \tau\left(\Phi\left(\left|\sum_{i=0}^n \lambda_i x_i\right|\right)\right) \leq \sum_{i=0}^n \lambda_i \tau\left(\Phi(|x_i|)\right).$$

We now recall the definition of Orlicz spaces. Given an Orlicz function Φ , the Orlicz function space $L_\Phi(0, \alpha)$, $0 < \alpha \leq \infty$ is the set of all measurable functions f on $(0, \alpha)$ such that

$$\|f\|_{L_\Phi} := \inf \left\{ \lambda > 0, \int_0^\infty \Phi\left(\frac{|f(t)|}{\lambda}\right) dt \leq 1 \right\}.$$

The Banach space $(L_\Phi(0, \alpha), \|\cdot\|_{L_\Phi})$ has the Fatou property (see [35, p.64]) and hence is a fully symmetric Banach space in the sense of [19]. An Orlicz function is called to satisfy Δ_2 -condition if there exists a constant $a > 1$ such that $\Phi(at) \lesssim_a \Phi(t)$ for all $t > 0$. Note that if Φ satisfies Δ_2 -condition, then $f \in L_\Phi(0, \alpha)$ if and only if $\int_0^\infty \Phi(|f|) < \infty$. If Φ satisfy the Δ_2 -condition, then by (2.3) we have

$$(2.4) \quad \tau(\Phi(x+y)) \lesssim_\Phi \tau(\Phi(x)) + \tau(\Phi(y)), \quad x, y \in L_\Phi(\mathcal{M}).$$

Given $1 \leq p \leq q \leq \infty$, an Orlicz function Φ is said to be p -convex if the function $t \rightarrow \Phi(t^{\frac{1}{p}})$, $t > 0$ is convex, and Φ is said to be q -concave if the function $t \rightarrow \Phi(t^{\frac{1}{q}})$, $t > 0$ is concave. If Φ is p -convex and q -concave for $1 \leq p \leq q \leq \infty$, then the associated Orlicz space L_Φ is p -convex and is q -concave [31, Theorem 50], and hence the Boyd indices of L_Φ satisfy $p \leq p_{L_\Phi} \leq q_{L_\Phi} \leq q$ (see [37] or [15, Lemma 4.9]), which also implies that the standard Matuzewska-Orlicz indices of the Orlicz function Φ satisfy $p \leq p_\Phi \leq q_\Phi \leq q$ [41, Theorem 4.2]. It also follows from [41, Theorem 3.2 (b)] that the Δ_2 -condition is equivalent to $q_\Phi < \infty$.

2.5. Hardy-Littlewood preorder. The following notion is used in the sequel.

Definition 2.1. Let \mathcal{M} be a semifinite von Neumann algebra and $x, y \in L_1(\mathcal{M}) + \mathcal{M}$. We write $y \prec\prec x$ if

$$\int_0^t \mu(s, y) ds \leq \int_0^t \mu(s, x) ds, \quad t > 0.$$

It is proved in [35, Theorem II.3.2] that $y \prec\prec x$ if and only if there exists a $T : L_1 + L_\infty \rightarrow L_1 + L_\infty$ such that $T : L_1 \rightarrow L_1$, $T : L_\infty \rightarrow L_\infty$ are contractions and such that $Tx = y$.

We say that E is fully symmetric if $\|\cdot\|_E$ is monotone with respect to $\prec\prec$. The following result is due to Calderón and Mityagin (see [35, Theorem II.3.4]).

Theorem 2.2. Symmetric quasi-Banach function space $E \in \text{Int}(L_1, L_\infty)$ iff it is fully symmetric.

It follows from [20, Theorem 11] that for every Orlicz function Φ

$$(2.5) \quad x \prec\prec y \implies \tau(\Phi(x)) \leq \tau(\Phi(y)), \quad \forall x, y \in L_1(\mathcal{M}) + \mathcal{M}.$$

If $x, y, z \in L_1(\mathcal{M}) + \mathcal{M}$, then $x \otimes z, y \otimes z \in L_1(\mathcal{M} \bar{\otimes} \mathcal{M}) + \mathcal{M} \bar{\otimes} \mathcal{M}$. It follows from above that if $x, y, z \in L_1(\mathcal{M}) + \mathcal{M}$, then

$$(2.6) \quad y \prec\prec x \implies y \otimes z \prec\prec x \otimes z.$$

2.6. Noncommutative martingales. A noncommutative probability space is a couple (\mathcal{M}, τ) where \mathcal{M} is a finite von Neumann algebra and τ is a normal faithful trace with $\tau(1) = 1$. Let $(\mathcal{M}_n)_{n \geq 0}$ be an increasing sequence of von Neumann subalgebras of \mathcal{M} such that the union of the \mathcal{M}_n 's is weak*-dense in \mathcal{M} . Let \mathcal{E}_n be the conditional expectation of \mathcal{M} with respect to \mathcal{M}_n .

Definition 2.3. A sequence $x = (x_k)_{k \geq 0}$ in $L_1(\mathcal{M})$ is a sequence of martingale differences if $x_k \in \mathcal{M}_k$ for $k \geq 0$ and if $\mathcal{E}_{k-1}(x_k) = 0$ for every $k \geq 1$.

In this paper, we always consider noncommutative martingales associated with a noncommutative probability space unless explicit explanation.

3. DISJOINTIFICATION INEQUALITIES IN SYMMETRIC OPERATOR SPACES

In this section we establish disjointification inequalities for noncommutative martingale difference sequences in symmetric operator spaces. We first state several lemmas.

Lemma 3.1. *Let (\mathcal{M}, τ) be a finite von Neumann algebra and let (\mathcal{N}, ν) be a semifinite atomless one. Suppose that $E \in \text{Int}(L_1(0, 1), L_2(0, 1))$ is a symmetric quasi-Banach space. If $T : L_2(\mathcal{N}) \rightarrow L_2(\mathcal{M})$ and $T : L_1(\mathcal{N}) \rightarrow L_1(\mathcal{M})$ is a linear contraction, then*

$$\|Tx\|_{E(\mathcal{M})} \lesssim_E \|x\|_{Z_E^2(\mathcal{N})}, \quad \forall x \in Z_E^2(\mathcal{N}).$$

Proof. To see this, fix $x = x^* \in Z_E^2(\mathcal{N})$ and consider the spectral projection of $|x|$ associated with the interval $(\mu(1, x), \infty)$,

$$p := e_{(\mu(1, x), \infty)}(|x|).$$

Clearly, $\nu(p) \leq 1$. Since \mathcal{N} is atomless, we can fix a projection $q \in \mathcal{N}$ such that $q \geq p$ and $\nu(q) = 1$. By assumptions, $T : L_1(q\mathcal{N}q) \rightarrow L_1(\mathcal{M})$ and $T : L_2(q\mathcal{N}q) \rightarrow L_2(\mathcal{M})$ is a contraction. Since E is an interpolation space between $L_1(0, 1)$ and $L_2(0, 1)$, it follows from [19, Theorem 3.2] that $T : E(q\mathcal{N}q) \rightarrow E(\mathcal{M})$ is a bounded map. Since $\mu(t, qxq) = 0$ for all t greater than trace of q , we have

$$\mu(qxq) = \mu(qxq)\chi_{(0,1)} \leq \mu(x)\chi_{(0,1)}.$$

On the other hand, $1 - q \leq 1 - p$ and therefore by (2.1),

$$\begin{aligned} \|qx(1 - q)\|_{L_2(\mathcal{N})}^2 &\leq \|x(1 - q)\|_{L_2(\mathcal{N})}^2 = \nu(x(1 - q)x) \leq \nu(x(1 - p)x) \\ &= \|x(1 - p)\|_{L_2(\mathcal{N})}^2 = \int_{\{\mu(x) \leq \mu(1, x)\}} \mu^2(t, x) dt \\ &\leq \|\mu(x)\chi_{(1, \infty)}\|_2^2 + \mu^2(1, x) \lesssim \|\mu(x)\|_{L_1 + L_2}^2. \end{aligned}$$

Similarly,

$$\|(1 - q)x\|_{L_2(\mathcal{N})}^2 \lesssim \|\mu(x)\|_{L_1 + L_2}^2.$$

Consequently, taking into account that $\|\cdot\|_E \leq \|\cdot\|_2$ we have

$$\begin{aligned} \|Tx\|_{E(\mathcal{M})} &\leq \|T(qxq)\|_{E(\mathcal{M})} + \|T(qx(1 - q))\|_{E(\mathcal{M})} + \|T((1 - q)x)\|_{E(\mathcal{M})} \\ &\leq \|T(qxq)\|_{E(\mathcal{M})} + \|T(qx(1 - q))\|_{L_2(\mathcal{M})} + \|T((1 - q)x)\|_{L_2(\mathcal{M})} \\ &\lesssim_E \|qxq\|_{E(q\mathcal{N}q)} + \|qx(1 - q)\|_{L_2(\mathcal{N})} + \|(1 - q)x\|_{L_2(\mathcal{N})} \\ &\lesssim \|\mu(x)\chi_{(0,1)}\|_E + \|\mu(x)\|_{L_1 + L_2} \lesssim \|x\|_{Z_E^2(\mathcal{N})}. \end{aligned}$$

Hence,

$$\|Tx\|_{E(\mathcal{M})} \leq \|T(\Re(x))\|_{E(\mathcal{M})} + \|T(\Im(x))\|_{E(\mathcal{M})} \lesssim_E \|x\|_{Z_E^2(\mathcal{N})}, \quad \forall x \in Z_E^2(\mathcal{N}).$$

□

The following proposition extends the result of [4, Lemma 3.5] to noncommutative setting. The assumption that E is a separable symmetric Banach space was used in [4].

Proposition 3.2. *Let $E \in \text{Int}(L_1(0, 1), L_2(0, 1))$ be a symmetric quasi-Banach space and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be a sequence of martingale differences. We have*

$$\left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \lesssim_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)}.$$

Proof. Without loss of generality, we may assume that \mathcal{M} is atomless. Indeed, otherwise, we consider algebra $\mathcal{M} \bar{\otimes} L_\infty(0, 1)$ and a sequence $\{x_k \otimes 1\}_{k \geq 0}$.

Consider the operator

$$T : \sum_{k \geq 0} x_k \otimes e_k \rightarrow \sum_{k \geq 0} (\mathcal{E}_k(x_k) - \mathcal{E}_{k-1}(x_k)), \quad \forall (x_k)_{k \geq 0} \subset (L_1 + L_2)(\mathcal{M}).$$

We claim that T is a bounded map from $L_2(\mathcal{M} \bar{\otimes} \ell_\infty)$ to $L_2(\mathcal{M})$. In fact $\{\mathcal{E}_k(x_k) - \mathcal{E}_{k-1}(x_k)\}_{k \geq 0}$ is an orthogonal sequence and, therefore,

$$\|T(\sum_{k \geq 0} x_k \otimes e_k)\|_2^2 = \sum_{k \geq 0} \|\mathcal{E}_k(x_k) - \mathcal{E}_{k-1}(x_k)\|_2^2 \leq 4 \|\sum_{k \geq 0} x_k \otimes e_k\|_2^2.$$

We also claim that $T : L_1(\mathcal{M} \bar{\otimes} \ell_\infty) \rightarrow L_1(\mathcal{M})$ is bounded. Indeed, we have

$$\|T(\sum_{k \geq 0} x_k \otimes e_k)\|_1 \leq \sum_{k \geq 0} \|\mathcal{E}_k(x_k) - \mathcal{E}_{k-1}(x_k)\|_1 \leq 2 \|\sum_{k \geq 0} x_k \otimes e_k\|_1.$$

Using Lemma 3.1, we obtain that $T : Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty) \rightarrow E(\mathcal{M})$ and, moreover,

$$\|Tx\|_{E(\mathcal{M})} \lesssim_E \|x\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)}, \quad x \in Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty).$$

In particular, if $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ is a sequence of martingale differences, then

$$\|\sum_{k \geq 0} x_k\|_{E(\mathcal{M})} = \|T(\sum_{k \geq 0} x_k \otimes e_k)\|_{E(\mathcal{M})} \lesssim_E \|\sum_{k \geq 0} x_k \otimes e_k\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)}.$$

This completes the proof. \square

Lemma 3.3. *Let (\mathcal{M}, τ) be a finite von Neumann algebra and let (\mathcal{N}, ν) be a semifinite atomless one. Let $E \in \text{Int}(L_2(0, 1), L_\infty(0, 1))$ be a symmetric quasi-Banach space. If $T : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{N})$ and $T : L_\infty(\mathcal{M}) \rightarrow L_\infty(\mathcal{N})$ is a linear contraction which maps self-adjoint operators to self-adjoint ones, then*

$$\|Tx\|_{Z_E^2(\mathcal{N})} \lesssim_E \|x\|_{E(\mathcal{M})}, \quad x \in E(\mathcal{M}).$$

Proof. To see this, fix $x = x^* \in E(\mathcal{M})$ and consider the spectral projection of $|Tx|$ associated with the interval $(\mu(1, Tx), \infty)$,

$$p := e_{(\mu(1, Tx), \infty)}(|Tx|).$$

It is clear that $\nu(p) \leq 1$. Fix a projection $q \in \mathcal{N}$ such that $q \geq p$ and $\nu(q) = 1$. Consider the operator

$$S : z \rightarrow q \cdot Tz \cdot q, \quad z \in (L_2 + L_\infty)(\mathcal{M}).$$

Using the assumption on T , we immediately obtain that $S : L_2(\mathcal{M}) \rightarrow L_2(q\mathcal{N}q)$ and $S : L_\infty(\mathcal{M}) \rightarrow L_\infty(q\mathcal{N}q)$ are also contractions. Since E is an interpolation space between $L_2(0, 1)$ and $L_\infty(0, 1)$, it follows from [19, Theorem 3.2] that $S : E(\mathcal{M}) \rightarrow E(q\mathcal{N}q)$ is a bounded map. Noting that $\mu(q \cdot Tx \cdot q) = \mu(q \cdot Tx \cdot q)\chi_{(0, 1)}$ and

$$\|(1 - q) \cdot Tx\|_{L_\infty(\mathcal{N})} \leq \|(1 - p) \cdot Tx\|_{L_\infty(\mathcal{N})} \leq \mu(1, Tx),$$

we have

$$\begin{aligned}
\|Tx\|_{Z_E^2(\mathcal{N})} &\leq \|q \cdot Tx \cdot q\|_{Z_E^2(\mathcal{N})} + \|q \cdot Tx \cdot (1-q)\|_{Z_E^2(\mathcal{N})} + \|(1-q) \cdot Tx\|_{Z_E^2(\mathcal{N})} \\
&\leq \|Sx\|_{E(q\mathcal{N}q)} + \|q \cdot Tx \cdot q\|_2 + \|q \cdot Tx \cdot (1-q)\|_{(L_2 \cap L_\infty)(\mathcal{N})} \\
&+ \|(1-q) \cdot Tx\|_{(L_2 \cap L_\infty)(\mathcal{N})} \leq \|Sx\|_{E(q\mathcal{N}q)} + 3\|Tx\|_{L_2(\mathcal{N})} \\
&+ 2\|(1-q) \cdot Tx\|_{L_\infty(\mathcal{N})} \lesssim_E \|Sx\|_{E(q\mathcal{N}q)} + 3\|Tx\|_{L_2(\mathcal{N})} + 2\mu(1, Tx) \\
&\leq \|Sx\|_{E(q\mathcal{N}q)} + 5\|Tx\|_{L_2(\mathcal{N})}.
\end{aligned}$$

Thus,

$$\|Tx\|_{Z_E^2(\mathcal{N})} \lesssim_E \|x\|_{E(\mathcal{M})} + 5\|x\|_{L_2(\mathcal{M})} \leq 6\|x\|_{E(\mathcal{M})}, \quad x = x^* \in E(\mathcal{M}).$$

By splitting x into its real part and imaginary parts, we conclude the proof. \square

Remark 3.4. Suppose that in Lemma 3.3, we have that $E \in \text{Int}(L_2(0, 1), L_q(0, 1))$ is a symmetric quasi-Banach space for some $2 < q < \infty$, and that $T : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{N})$ and $T : L_q(\mathcal{M}) \rightarrow L_q(\mathcal{N})$ are linear contractions which map self-adjoint operators to self-adjoint ones. In this case, the result of Lemma 3.3 still holds. Indeed, we just need to notice that

$$\begin{aligned}
\|(1-q) \cdot Tx\|_{L_q(\mathcal{N})} &\leq \|(1-p) \cdot Tx\|_{L_q(\mathcal{N})} \\
&\leq \left(\int_1^\infty \mu^q(t, Tx) dt \right)^{1/q} + \mu(1, Tx) \\
&\lesssim \|Tx\|_{(L_2 + L_q)(\mathcal{N})} \leq \|x\|_{(L_2 + L_q)(\mathcal{M})} \\
&= \|x\|_{L_2(\mathcal{M})} \leq \|x\|_{E(\mathcal{M})}.
\end{aligned}$$

The following proposition extends Theorem 3.5 and [6, Corollary 3.6].

Proposition 3.5. Let $E \in \text{Int}(L_2(0, 1), L_\infty(0, 1))$ be a symmetric quasi-Banach space and let $\{x_k\}_{k \geq 0} \subset E(\mathcal{M})$ be a sequence of martingale differences. We have

$$\left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)} \lesssim_E \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})}.$$

Proof. Let $\{\mathcal{M}_k\}_{k \geq 0} \subset \mathcal{M}$ be an increasing sequence of unital von Neumann subalgebras and let $\mathcal{E}_k : \mathcal{M} \rightarrow \mathcal{M}_k$ be the conditional expectations. For brevity, we set $\mathcal{E}_{-1} = 0$. Consider the operator

$$T : x \rightarrow \sum_{k \geq 0} \left(\mathcal{E}_k(x) - \mathcal{E}_{k-1}(x) \right) \otimes e_k, \quad x \in L_2(\mathcal{M}).$$

We have $T : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M} \bar{\otimes} \ell_\infty)$ (in fact, $\|Tx\|_2 = \|x\|_2$ for every $x \in L_2(\mathcal{M})$). It is immediate that $\|Tx\|_\infty \leq 2\|x\|_\infty$. It follows from Lemma 3.3 that T boundedly maps $E(\mathcal{M})$ into $Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)$. Applying this result to the element $x = \sum_{k \geq 0} x_k$, we have

$$Tx = \sum_{k \geq 0} \left(\mathcal{E}_k(x) - \mathcal{E}_{k-1}(x) \right) \otimes e_k = \sum_{k \geq 0} x_k \otimes e_k$$

and the assertion immediately follows. \square

Our first main result Theorem 1.1 now follows immediately from the combination of Proposition 3.2 and Proposition 3.5.

Corollary 3.6. Let E be a symmetric quasi-Banach space on $(0, 1)$. Let $(x_k)_{k=0}^\infty \subset E(\mathcal{M})$ be an arbitrary sequence and $(r_k)_{k=0}^\infty$ be the Rademacher sequence on $(0, 1)$.

(i) If $E \in \text{Int}(L_1(0, 1), L_2(0, 1))$, then

$$\left\| \sum_{k \geq 0} x_k \otimes r_k \right\|_{E(\mathcal{M} \bar{\otimes} L_\infty(0,1))} \lesssim_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)}.$$

(ii) If $E \in \text{Int}(L_2(0, 1), L_\infty(0, 1))$, then

$$\left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)} \lesssim_E \left\| \sum_{k \geq 0} x_k \otimes r_k \right\|_{E(\mathcal{M} \bar{\otimes} L_\infty(0,1))}.$$

We now prove Theorem 1.2. Let E be a symmetric quasi-Banach space. For $s > 0$, define $\sigma_s : E \rightarrow E$ by setting (see [35])

$$\sigma_s f(t) = f(t/s), \quad t > 0, \quad f \in E.$$

Proof of Theorem 1.2. Recall that the classical Haar system $(h_k)_{k \geq 0}$ is defined as follows. First, set $h_0 = 1$. If $k = 2^m + l$, $0 \leq l < 2^m$, then

$$h_k = \chi_{(\frac{2l}{2^m+1}, \frac{2l+1}{2^m+1})} - \chi_{(\frac{2l+1}{2^m+1}, \frac{2l+2}{2^m+1})}.$$

Define a filtration³ $(\mathcal{M}_k)_{k \geq 0}$ in $L_\infty(0, 1)$ as follows. First, set $\mathcal{M}_0 = \mathbb{C}$. If $k = 2^m + l$, $0 \leq l < 2^m$, then

$$\mathcal{M}_k = \text{span}\left\{ \left(\chi_{(\frac{j}{2^m+1}, \frac{j+1}{2^m+1})} \right)_{j=0}^{2l+1}, \left(\chi_{(\frac{j}{2^m}, \frac{j+1}{2^m})} \right)_{j=l+1}^{2^m-1} \right\}$$

It is immediate that $(h_k)_{k \geq 0}$ is a sequence of martingale differences in $L_\infty(0, 1)$ with respect to the filtration $(\mathcal{M}_k)_{k \geq 0}$.

Now we apply equality (1.5) to the sequences $(\alpha_k h_k)_{k \geq 0}$ and $(|\alpha_k| h_k)_{k \geq 0}$ of martingale differences (here, $(\alpha_k)_{k \geq 0}$ is a sequence of scalars). It is immediate that

$$\left\| \sum_{k \geq 0} \alpha_k h_k \right\|_{E(0,1)} \approx_E \left\| \sum_{k \geq 0} \alpha_k h_k \otimes e_k \right\|_{Z_E^2(L_\infty(0,1) \bar{\otimes} \ell_\infty)} \approx_E \left\| \sum_{k \geq 0} |\alpha_k| h_k \right\|_{E(0,1)}.$$

Thus, Haar system is unconditional in E . According to [37, Theorem 2.c.6], we have that $E \in \text{Int}(L_p, L_q)$ for some $1 < p < q < \infty$.

Now, fix $x \in L_\infty(0, 1)$ and consider the sequence $(x \otimes r_k)_{k=0}^{n-1}$. Since the latter sequence consists of martingale differences, it follows from (1.5) that

$$\left\| x \otimes \sum_{k=0}^{n-1} r_k \right\|_E \approx_E \left\| \sigma_n \mu(x) \right\|_{Z_E^2} \approx \left\| (\sigma_n \mu(x)) \chi_{(0,1)} \right\|_E + \left\| (\sigma_n \mu(x)) \chi_{(1,\infty)} \right\|_2,$$

where the first " \approx " follows from a simple fact that

$$\mu \left(\bigoplus_{k=0}^{n-1} (x \otimes r_k) \right) = \sigma_n \mu(x).$$

Note that

$$\left\| (\sigma_n \mu(x)) \chi_{(0,1)} \right\|_E \leq \|x\|_\infty \quad \text{and} \quad \left\| (\sigma_n \mu(x)) \chi_{(1,\infty)} \right\|_2 = n^{1/2} \left(\int_{\frac{1}{n}}^\infty \mu^2(t, x) dt \right)^{1/2}.$$

Dividing by $n^{1/2}$, setting $x_n = \frac{1}{n^{1/2}} \sum_{k=0}^{n-1} r_k$ and passing $n \rightarrow \infty$, we obtain that

$$\limsup_{n \rightarrow \infty} \|x \otimes x_n\|_E \approx_E \|x\|_2.$$

³We took the filtration from the Proposition 6.1.3 in [1].

Since for every $0 < t \leq 1$,

$$\|x_n\|_1 = \frac{1}{t} \int_0^t \mu\left(\frac{s}{t}, x_n\right) ds \leq \frac{1}{t} \int_0^t \mu(s, x_n) ds,$$

we get $\|x_n\|_1 \prec\prec x_n$, and hence by (2.6),

$$x \otimes \|x_n\|_1 \prec\prec x \otimes x_n \quad \text{and} \quad \|x \otimes \|x_n\|_1\|_E \lesssim_E \|x \otimes x_n\|_E.$$

Therefore, by Khinchine inequality for L_1 ,

$$(3.1) \quad \|x\|_E \approx \limsup_{n \rightarrow \infty} \|x\|_E \|x_n\|_1 \leq \limsup_{n \rightarrow \infty} \|x \otimes x_n\|_E \approx_E \|x\|_2.$$

On the other hand, the operators $x \rightarrow x \otimes x_n$ are uniformly bounded in L_1 and in L_q (by Khinchine inequality). Hence, these operators are uniformly bounded in E because that $E \in \text{Int}(L_1, L_q)$. Thus,

$$(3.2) \quad \|x\|_2 \approx_E \limsup_{n \rightarrow \infty} \|x \otimes x_n\|_E \lesssim_q \|x\|_E.$$

Combining (3.1) and (3.2), we get

$$(3.3) \quad \|x\|_2 \approx_E \|x\|_E, \quad x \in L_\infty(0, 1).$$

We now can omit the assumption $x \in L_\infty(0, 1)$. Firstly for every $x \in E$, it follows from (3.3) that

$$\|x\|_E \geq \|\mu(x)\chi_{(\frac{1}{n}, 1)}\|_E \approx_E \|\mu(x)\chi_{(\frac{1}{n}, 1)}\|_2, \quad \forall n \geq 1.$$

Passing $n \rightarrow \infty$, we get $\|x\|_2 \lesssim \|x\|_E$ and, therefore, $E \subset L_2$. If $x \in L_2$, then there exists $(x_k) \subset L_\infty(0, 1)$ such that $\|x_k - x\|_2 \rightarrow 0$. Therefore, again by (3.3)

$$\|x_k - x_l\|_E \approx_E \|x_k - x_l\|_2 \rightarrow 0, \quad k, l \rightarrow \infty,$$

which implies that $x_k \rightarrow x$ in E . Thus, $x \in E$ and we conclude the proof. \square

4. BURKHOLDER-GUNDY INEQUALITY IN SYMMETRIC BANACH SPACES

The main result in this section is Theorem 1.3, which extends the Pisier-Xu noncommutative Burkholder-Gundy inequality [44, Theorem 2.1] and also improves [24, Theorem 3.1] and [17, Proposition 4.18]. We first state several lemmas.

Lemma 4.1. *Let E be a symmetric Banach space on $(0, 1)$ and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be an arbitrary sequence. If $E \in \text{Int}(L_1(0, 1), L_2(0, 1))$, then*

$$\left\| \sum_{k \geq 0} x_k \otimes r_k \right\|_{E(\mathcal{M} \bar{\otimes} L_\infty(0, 1))} \lesssim_E \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

Proof. Without loss of generality, we may assume that \mathcal{M} is atomless. Indeed, otherwise, we consider algebra $\mathcal{M} \bar{\otimes} L_\infty(0, 1)$ and a sequence $\{x_k \otimes 1\}_{k \geq 0}$.

Recalling that the column subspace of $L_p(\mathcal{M} \bar{\otimes} B(\ell_2))$ is a 1-complemented subspace for every $0 < p \leq \infty$, we consider the following linear mapping defined on $(L_1 + L_2)(\mathcal{M} \bar{\otimes} B(\ell_2))$

$$T : \sum_{k \geq 0} x_k \otimes e_{k1} \rightarrow \sum_{k \geq 0} x_k \otimes r_k.$$

Since for every $x = \sum_{i,j \geq 0} x_{ij} \otimes e_{ij} \in L_2(\mathcal{M} \bar{\otimes} B(\ell_2))$,

$$\left\| \sum_{i \geq 0} x_{i1} \otimes r_i \right\|_{L_2(\mathcal{M} \bar{\otimes} L_\infty(0, 1))} = \left\| \sum_{i \geq 0} x_{i1} \otimes e_{i1} \right\|_2 = \left\| \sum_{i,j \geq 0} (x_{ij} \otimes e_{ij})(1 \otimes e_{11}) \right\|_2 \leq \|x\|_2.$$

We now have a linear bounded operator $S : L_2(\mathcal{M} \bar{\otimes} B(\ell_2)) \rightarrow L_2(\mathcal{M} \bar{\otimes} L_\infty(0, 1))$ defined by the setting

$$Sx = T(x \cdot (1 \otimes e_{11})), \quad x \in L_2(\mathcal{M} \bar{\otimes} B(\ell_2)).$$

It follows from noncommutative Khinchine inequality for $L_1(\mathcal{M})$ (see [40]) that for any $x = \sum_{i,j \geq 0} x_{ij} \otimes e_{ij} \in L_1(\mathcal{M} \bar{\otimes} B(\ell_2))$,

$$\|Sx\|_{L_1(\mathcal{M} \bar{\otimes} L_\infty(0,1))} \leq \|(\sum_{k \geq 0} |x_{k1}|^2)^{1/2}\|_{L_1(\mathcal{M})} \leq \|\sum_{i \geq 0} x_{i1} \otimes e_{i1}\|_1 \leq \|x\|_1.$$

Applying Lemma 3.1, we obtain

$$\|Sx\|_{E(\mathcal{M} \bar{\otimes} L_\infty(0,1))} \lesssim_E \|x\|_{Z_E^2(\mathcal{M} \bar{\otimes} B(\ell_2))}.$$

Setting $x = \sum_{k \geq 0} x_k \otimes e_{k1}$, we get

$$\|\sum_{k \geq 0} x_k \otimes r_k\|_{E(\mathcal{M} \bar{\otimes} L_\infty(0,1))} \lesssim_E \|\sum_{k \geq 0} x_k \otimes e_{k1}\|_{Z_E^2(\mathcal{M} \bar{\otimes} B(\ell_2))}.$$

However,

$$|\sum_{k \geq 0} x_k \otimes e_{k1}| = \left(\sum_{k \geq 0} |x_k|^2\right)^{1/2} \otimes e_{11}$$

and, therefore,

$$\begin{aligned} \|\sum_{k \geq 0} x_k \otimes r_k\|_{E(\mathcal{M} \bar{\otimes} L_\infty(0,1))} &\lesssim_E \left\| \left(\sum_{k \geq 0} |x_k|^2\right)^{1/2} \otimes e_{11} \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} B(\ell_2))} \\ &= \left\| \left(\sum_{k \geq 0} |x_k|^2\right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |x_k|^2\right)^{1/2} \right\|_{(L_1+L_2)(\mathcal{M})} \\ &\leq 2 \left\| \left(\sum_{k \geq 0} |x_k|^2\right)^{1/2} \right\|_{E(\mathcal{M})}. \end{aligned}$$

□

Lemma 4.2. *Let E be a symmetric Banach space on $(0, 1)$ and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be a sequence of martingale differences. If $E \in \text{Int}(L_2(0, 1), L_q(0, 1))$ for some $2 < q < \infty$, then*

$$\left\| \left(\sum_{k \geq 0} |x_k|^2\right)^{1/2} \right\|_{E(\mathcal{M})} \lesssim_E \|\sum_{k \geq 0} x_k\|_{E(\mathcal{M})}.$$

Proof. Consider the linear mapping

$$T : x \rightarrow \sum_{k \geq 0} (\mathcal{E}_k(x) - \mathcal{E}_{k-1}(x)) \otimes e_{k1}, \quad x \in L_q(\mathcal{M}).$$

Clearly, $T : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M} \bar{\otimes} B(\ell_2))$ (moreover, $\|Tx\|_{L_2(\mathcal{M} \bar{\otimes} B(\ell_2))} = \|x\|_{L_2(\mathcal{M})}$). Using [44, Theorem 2.1] (see also [46, Theorem 5.1]), we obtain

$$\|Tx\|_{L_q(\mathcal{M} \bar{\otimes} B(\ell_2))} = \left\| \left(\sum_{k \geq 0} |\mathcal{E}_k(x) - \mathcal{E}_{k-1}(x)|^2\right)^{1/2} \right\|_{L_q(\mathcal{M})} \lesssim_q \|x\|_{L_q(\mathcal{M})}.$$

By Lemma 3.3 and Remark 3.4, we have that $\|Tx\|_{Z_E^2(\mathcal{M} \bar{\otimes} B(\ell_2))} \lesssim_E \|x\|_{E(\mathcal{M})}$. Again,

$$\left| \sum_{k \geq 0} (\mathcal{E}_k(x) - \mathcal{E}_{k-1}(x)) \otimes e_{k1} \right| = \left(\sum_{k \geq 0} |\mathcal{E}_k(x) - \mathcal{E}_{k-1}(x)|^2 \right)^{1/2} \otimes e_{11},$$

therefore,

$$\left\| \left(\sum_{k \geq 0} |\mathcal{E}_k(x) - \mathcal{E}_{k-1}(x)|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \lesssim_E \|x\|_{E(\mathcal{M})}.$$

Substituting $x = \sum_{k \geq 0} x_k$, we conclude the proof. \square

The following lemma is attributed in [17] to Bekjan [8]. However, p -convexity and/or q -concavity assumed everywhere in [8] makes result of that paper unsuitable for our purposes. We present a different proof based on the results from [24].

Lemma 4.3. *Let E be a symmetric Banach space on $(0, 1)$ and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be a sequence of martingale differences. If $E \in \text{Int}(L_p, L_2)$ for $1 < p < 2$, then*

$$\begin{aligned} & \inf_{x_k = y_k + z_k} \left(\left\| \left(\sum_{k \geq 0} |y_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |z_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right) \\ & \approx_E \inf_{\substack{x_k = y_k + z_k \\ \{y_k\}_{k \geq 0}, \{z_k\}_{k \geq 0} \text{ are} \\ \text{martingale differences}}} \left(\left\| \left(\sum_{k \geq 0} |y_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |z_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right). \end{aligned}$$

Proof. Let $x_k = y_k + z_k$. Since $(x_k)_{k \geq 0}$ is a sequence of martingale differences, it follows that

$$x_k = \mathcal{E}_k(x_k) - \mathcal{E}_{k-1}(x_k) = (\mathcal{E}_k(y_k) - \mathcal{E}_{k-1}(y_k)) + (\mathcal{E}_k(z_k) - \mathcal{E}_{k-1}(z_k)) \stackrel{\text{def}}{=} u_k + v_k.$$

Clearly, $(u_k)_{k \geq 0}$ and $(v_k)_{k \geq 0}$ are sequences of martingale differences. We have

$$\left\| \left(\sum_{k \geq 0} |u_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \leq \left\| \left(\sum_{k \geq 0} |\mathcal{E}_k(y_k)|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |\mathcal{E}_{k-1}(y_k)|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

It follows now from the noncommutative Stein inequality (see [24, Lemma 3.3]) that

$$\left\| \left(\sum_{k \geq 0} |u_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \lesssim_E \left\| \left(\sum_{k \geq 0} |y_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

Similarly, we have

$$\left\| \left(\sum_{k \geq 0} |v_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \lesssim_E \left\| \left(\sum_{k \geq 0} |z_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

Hence, the infimum can be taken over all the sequences of martingale differences. \square

We are now prepared to present the proof of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Since $E \in \text{Int}(L_p, L_q)$ for $1 < p < q < \infty$, it follows from [17, Lemma 4.17] that

$$(4.1) \quad \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \approx_E \left\| \sum_{k \geq 0} x_k \otimes r_k \right\|_{E(\mathcal{M} \bar{\otimes} L_\infty(0,1))}.$$

(i) [43, Theorem 1.1] states that if E is separable with finite upper Boyd index, then we have

$$\left\| \sum_{k \geq 0} x_k \otimes r_k \right\|_{E(\mathcal{M} \otimes L_\infty(0,1))} \gtrsim_E \inf_{x_k = y_k + z_k} \left(\left\| \left(\sum_{k \geq 0} |y_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |z_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right).$$

It immediately follows from (4.1) that

$$\left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \gtrsim_E \inf_{x_k = y_k + z_k} \left(\left\| \left(\sum_{k \geq 0} |y_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |z_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right),$$

where the infimum taken over all possible decompositions $x_k = y_k + z_k$. By Lemma 4.3, we have

$$\left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \gtrsim_E \inf_{x_k = y_k + z_k} \left(\left\| \left(\sum_{k \geq 0} |y_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |z_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right),$$

where the infimum is taken over the sequences of martingale differences. We now prove the converse inequality. Let $x_k = y_k + z_k$, with $(y_k)_{k \geq 0} \subset E(\mathcal{M})$ and $(z_k)_{k \geq 0} \subset E(\mathcal{M})$ being sequences of martingale differences. Applying (4.1) and Lemma 4.1, we conclude that

$$\left\| \sum_{k \geq 0} y_k \right\|_{E(\mathcal{M})} \lesssim_E \left\| \left(\sum_{k \geq 0} |y_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}$$

and, similarly,

$$\left\| \sum_{k \geq 0} z_k \right\|_{E(\mathcal{M})} \lesssim_E \left\| \left(\sum_{k \geq 0} |z_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

Therefore, we have

$$\left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \lesssim_E \inf_{x_k = y_k + z_k} \left(\left\| \left(\sum_{k \geq 0} |y_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |z_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right),$$

where the infimum is taken over the sequences of martingale differences.

(ii) [17, Theorem 4.1] states that if E is p -convex for some $0 < p < \infty$ (we take $p = 1$ since every Banach space is 1-convex) and has finite upper Boyd index, then we have

$$\left\| \sum_{k \geq 0} x_k \otimes r_k \right\|_{E(\mathcal{M} \otimes L_\infty(0,1))} \lesssim_E \max \left\{ \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}, \left\| \left(\sum_{k \geq 0} |x_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right\}.$$

Now it immediately follows from (4.1) that

$$\left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \lesssim_E \max \left\{ \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}, \left\| \left(\sum_{k \geq 0} |x_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right\}.$$

The inequality

$$\left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} \gtrsim_E \max \left\{ \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}, \left\| \left(\sum_{k \geq 0} |x_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right\}.$$

follows from Lemma 4.2. \square

Proof of Theorem 1.4. Let the Haar system $(h_k)_{k \geq 0}$ and filtration $(\mathcal{M}_k)_{k \geq 0}$ in $L_\infty(0, 1)$ be defined as in the proof of Theorem 1.2. Recall that $(h_k)_{k \geq 0}$ is a sequence of martingale differences in $L_\infty(0, 1)$ with respect to the filtration $(\mathcal{M}_k)_{k \geq 0}$.

Now we apply the equality (1.6) (or (1.7)) to the sequences $(\alpha_k h_k)_{k \geq 0}$ and $(|\alpha_k| h_k)_{k \geq 0}$ of martingale differences (here, $(\alpha_k)_{k \geq 0}$ is a sequence of scalars). It is immediate that

$$\left\| \sum_{k \geq 0} \alpha_k h_k \right\|_E \approx_E \left\| \left(\sum_{k \geq 0} |\alpha_k|^2 |h_k|^2 \right)^{1/2} \right\|_E \approx_E \left\| \sum_{k \geq 0} |\alpha_k| h_k \right\|_E.$$

Thus, Haar system is an unconditional basic sequence in E . According to [37, Theorem 2.c.6], we have that $E \in \text{Int}(L_p, L_q)$ for $1 < p < q < \infty$. \square

We refer to [29] for the maximal functions of noncommutative martingales and to [16] for the notation $E(\mathcal{M}; \ell_\infty)$. The assertion below follows from Theorem 1.3 above and [16, Theorem 5.7]. Its dual version is also true for the case $E \in \text{Int}(L_p(0, 1), L_2(0, 1))$ for some $1 < p < 2$. These results improve [16, Theorem 6.1].

Corollary 4.4. *Let E be a symmetric Banach space on $(0, 1)$ and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be a sequence of martingale differences. If $E \in \text{Int}(L_2(0, 1), L_q(0, 1))$ for some $2 < q < \infty$, then*

$$\left\| \left(\sum_{l=0}^k x_l \right)_{k \geq 0} \right\|_{E(\mathcal{M}; \ell_\infty)} \approx_E \max \left\{ \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}, \left\| \left(\sum_{k \geq 0} |x_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \right\}.$$

5. BURKHOLDER INEQUALITY IN SYMMETRIC BANACH SPACES

Applying Theorem 1.1 and Theorem 1.3 proved in the preceding sections, in this section we partly resolve one problem stated in [48]. We refer to [38, Theorem 3.3.6] for the following lemma.

Lemma 5.1. *Let \mathcal{M} be a semifinite von Neumann algebra and let $0 \leq A, B \in (L_1 + L_\infty)(\mathcal{M})$. Then $B \prec\prec A$ if and only if*

$$\tau((B - t)_+) \leq \tau((A - t)_+), \quad \forall t > 0.$$

Corollary 5.2. *Let \mathcal{M} be a semifinite von Neumann algebra and let $A_k, B_k \in (L_1 + L_\infty)(\mathcal{M})$, $k \geq 0$. If $B_k \prec\prec A_k$ for all $k \geq 0$, then*

$$\sum_{k \geq 0} B_k \otimes e_k \prec\prec \sum_{k \geq 0} A_k \otimes e_k.$$

Proof. Without loss of generality, $A_k, B_k \geq 0$. It follows from the assumption and Lemma 5.1 that

$$\tau((B_k - t)_+) \leq \tau((A_k - t)_+), \quad t > 0,$$

for every $k \geq 0$. Thus,⁴

$$\begin{aligned} (\tau \otimes \#)((\sum_{k \geq 0} B_k \otimes e_k - t)_+) &= \sum_{k \geq 0} \tau((B_k - t)_+) \leq \sum_{k \geq 0} \tau((A_k - t)_+) \\ &= (\tau \otimes \#)((\sum_{k \geq 0} A_k \otimes e_k - t)_+), \quad t > 0. \end{aligned}$$

Again applying Lemma 5.1, we conclude the proof. \square

⁴Here, $\#$ denotes the counting measure on \mathbb{Z}_+ , so that $\tau \otimes \#$ is a trace on the von Neumann algebra $\mathcal{M} \bar{\otimes} l_\infty$.

The following lemma is taken from [42, Theorem 1].

Lemma 5.3. *Let E and F be two symmetric quasi-Banach function spaces and $1 < p < \infty$. We have*

$$(E + F)^{(p)} = E^{(p)} + F^{(p)}, \quad \text{and} \quad (E + F)_{(p)} = E_{(p)} + F_{(p)}.$$

The following proposition contains crucial technical estimate needed for the proof of Theorem 1.5.

Proposition 5.4. *Let $E \in \text{Int}(L_2, L_4)$ be a symmetric quasi-Banach space and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be a sequence of martingale differences. We have*

$$\left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \lesssim_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_{E(2)}^2(\mathcal{M} \bar{\otimes} \ell_\infty)} + \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

Proof. If $E \in \text{Int}(L_2, L_4)$, then it follows from (2.2) that $E_{(2)} \in \text{Int}(L_1, L_2)$. We have

$$\begin{aligned} \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} &= \left\| \sum_{k \geq 0} |x_k|^2 \right\|_{E_{(2)}(\mathcal{M})}^{1/2} \\ &= \left\| \sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) + \sum_{k \geq 0} |x_k|^2 - \mathcal{E}_{k-1}(|x_k|^2) \right\|_{E_{(2)}(\mathcal{M})}^{1/2}. \end{aligned}$$

By the quasi-triangle inequality in $E_{(2)}(\mathcal{M})$, we have that

$$\left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \lesssim_E \left\| \sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right\|_{E_{(2)}(\mathcal{M})}^{1/2} + \left\| \sum_{k \geq 0} |x_k|^2 - \mathcal{E}_{k-1}(|x_k|^2) \right\|_{E_{(2)}(\mathcal{M})}^{1/2}.$$

Since $\{|x_k|^2 - \mathcal{E}_{k-1}(|x_k|^2)\}_{k \geq 0}$ is a sequence of martingale differences, it follows from Theorem 1.1 (i) that

$$\begin{aligned} (5.1) \quad \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} &\lesssim_E \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} \\ &\quad + \left\| \sum_{k \geq 0} \left(|x_k|^2 - \mathcal{E}_{k-1}(|x_k|^2) \right) \otimes e_k \right\|_{Z_{E(2)}^2(\mathcal{M} \bar{\otimes} \ell_\infty)}^{1/2}. \end{aligned}$$

Since every conditional expectation operator is a contraction on $L_1(0, 1)$ and $L_\infty(0, 1)$, it follows from [35, Theorem II.3.4] (see also [38, Lemma 3.6.2]) that

$$\mathcal{E}_{k-1}(|x_k|^2) \prec\prec |x_k|^2, \quad \forall k \geq 0,$$

and therefore, by Corollary 5.2 we obtain

$$\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \otimes e_k \prec\prec \sum_{k \geq 0} |x_k|^2 \otimes e_k.$$

Since $E_{(2)} \in \text{Int}(L_1, L_2)$, it follows that $E_{(2)} \in \text{Int}(L_1, L_\infty)$. That is, up to a constant, $E_{(2)}$ (and, hence, $Z_{E(2)}^2$) is a fully symmetric quasi-Banach space. Thus,

$$(5.2) \quad \left\| \sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \otimes e_k \right\|_{Z_{E(2)}^2(\mathcal{M} \bar{\otimes} \ell_\infty)} \lesssim_E \left\| \sum_{k \geq 0} |x_k|^2 \otimes e_k \right\|_{Z_{E(2)}^2(\mathcal{M} \bar{\otimes} \ell_\infty)}.$$

Using quasi-triangle inequality, and combining (5.1) and (5.2), we deduce that

$$(5.3) \quad \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \lesssim_E \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \sum_{k \geq 0} |x_k|^2 \otimes e_k \right\|_{Z_{E(2)}^2(\mathcal{M} \bar{\otimes} \ell_\infty)}^{1/2}.$$

Let $X := \sum_{k \geq 0} |x_k| \otimes e_k$. It now remains to verify that

$$(5.4) \quad \|X^2\|_{Z_{E(2)}^2(\mathcal{M} \bar{\otimes} \ell_\infty)}^{1/2} \lesssim_E \|X\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)}.$$

Indeed, it follows from the definition of $\|\cdot\|_{Z_{E(2)}^2}$, Lemma 5.3 and (2.1) that

$$\begin{aligned} \|X^2\|_{Z_{E(2)}^2}^{1/2} &\approx \|\mu(X^2)\chi_{(0,1)}\|_{E(2)}^{1/2} + \|X^2\|_{L_1+L_2}^{1/2} = \|\mu(X)\chi_{(0,1)}\|_E + \|X\|_{(L_1+L_2)^{(2)}} \\ &\approx \|\mu(X)\chi_{(0,1)}\|_E + \|X\|_{L_2+L_4} \leq \|\mu(X)\chi_{(0,1)}\|_E + \|X\|_2 \\ &\leq 2\|\mu(X)\chi_{(0,1)}\|_E + \|\mu(X)\chi_{[1,\infty)}\|_2 \\ &\lesssim \|\mu(X)\chi_{(0,1)}\|_E + \|\mu(X)\|_{L_1+L_2} = \|X\|_{Z_E^2}. \end{aligned}$$

Combining (5.3) and (5.4), we complete the proof. \square

Proposition 5.5. *Let $E \in \text{Int}(L_2(0,1), L_4(0,1))$ be a symmetric quasi-Banach space and let $x = (x_k)_{k \geq 0}$ be a sequence of martingale differences. We have*

$$\begin{aligned} \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} &\lesssim_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)} \\ &+ \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k^*|^2) \right)^{1/2} \right\|_{E(\mathcal{M})}. \end{aligned}$$

Proof. It follows from Theorem 1.3 (ii) that

$$\left\| \sum_k x_k \right\|_{E(\mathcal{M})} \lesssim_{E(\mathcal{M})} \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |x_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

By Proposition 5.4, we have

$$\left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \lesssim_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)} + \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

Similarly,

$$\left\| \left(\sum_{k \geq 0} |x_k^*|^2 \right)^{1/2} \right\|_E \lesssim_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)} + \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k^*|^2) \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

Combining the preceding estimates, we complete the proof. \square

In order to prove the converse inequality in Proposition 5.5, we use a crucially important result due to Junge [29]. For convenience of the reader, we present a short proof in the appendix.

Theorem 5.6. *Let \mathcal{M} be a semifinite von Neumann algebra and let \mathcal{N} be a von Neumann subalgebra of \mathcal{M} which admits a normal conditional expectation $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$. There exists a linear isometry $u : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{N} \bar{\otimes} \mathcal{B}(\ell_2))$ such that*

$$u(x)^* u(y) = \mathcal{E}(x^* y) \otimes e_{11}, \quad \forall x, y \in L_2(\mathcal{M}).$$

Proposition 5.7. *Let $E \in \text{Int}(L_2(0,1), L_q(0,1))$, $2 < q < \infty$, be a symmetric quasi-Banach function space and let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be an arbitrary sequence. We have*

$$\left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} \lesssim_E \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

Proof. We set $E(0, \infty) = \{f \in L_1(0, \infty) + L_\infty(0, \infty) : \mu(f) \in E(0, 1)\}$ equipped with norm $\|f\|_{E(0, \infty)} := \|\mu(f)\|_{E(0, 1)}$. Then $(E(0, \infty), \|\cdot\|_{E(0, \infty)})$ is a quasi-Banach space. By Theorem 5.6, it is easy to see for $2 \leq p \leq \infty$,

$$\|u(x)\|_{L_p(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))} = \|\mathcal{E}(|x|^2)\|_{L_{\frac{p}{2}}(\mathcal{M})}^{1/2} \leq \|x\|_{L_p(\mathcal{M})}, \quad x \in L_2(\mathcal{M}) \cap L_p(\mathcal{M}).$$

By density, u is bounded in $L_p(\mathcal{M})$ for $2 \leq p \leq \infty$, and hence u is well defined on $E(\mathcal{M})$. Now it follows that for an arbitrary sequence $(y_k)_{k \geq 0} \subset E(\mathcal{M})$

$$\begin{aligned} \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|y_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} &= \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|y_k|^2) \otimes e_{11} \right)^{1/2} \right\|_{E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))} \\ &= \left\| \left(\sum_{k \geq 0} u_{k-1}(y_k)^* u_{k-1}(y_k) \right)^{1/2} \right\|_{E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))} \\ &= \left\| \sum_{k \geq 0} u_{k-1}(y_k) \otimes e_{k1} \right\|_{E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{B}(\ell_2))}. \end{aligned}$$

Let $2 \leq r < \infty$. Consider the mapping $S : L_r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)) \rightarrow L_r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{B}(\ell_2))$ by setting

$$Sy = T(y \cdot (1 \otimes e_{11})), \quad y = \sum_{k,j} y_{kj} \otimes e_{kj} \in L_r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)),$$

where

$$T : \sum_{k \geq 0} y_{k1} \otimes e_{k1} \rightarrow \sum_{k \geq 0} u_k(y_{k1}) \otimes e_{k1}.$$

We claim that S is bounded from $L_r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))$ into $L_r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{B}(\ell_2))$. Indeed, by the dual Doob inequality [29, Theorem 0.1] we have

$$\begin{aligned} \|Sy\|_{L_r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{B}(\ell_2))} &= \left\| \sum_{k \geq 0} u_{k-1}(y_{k1}) \otimes e_{k1} \right\|_{L_r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{B}(\ell_2))} \\ &= \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|y_{k1}|^2) \right)^{1/2} \right\|_{L_r(\mathcal{M})} \\ &= \left\| \sum_{k \geq 0} \mathcal{E}_{k-1}(|y_{k1}|^2) \right\|_{L_{\frac{r}{2}}(\mathcal{M})}^{1/2} \\ &\leq \left\| \sum_{k \geq 0} |y_{k1}|^2 \right\|_{L_{\frac{r}{2}}(\mathcal{M})}^{1/2} = \left\| \left(\sum_{k \geq 0} |y_{k1}|^2 \right)^{1/2} \right\|_{L_r(\mathcal{M})} \\ &= \left\| \sum_{k \geq 0} y_{k1} \otimes e_{k1} \right\|_{L_r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))} \leq \|y\|_{L_r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))}. \end{aligned}$$

Hence, S maps boundedly from $E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))$ into $E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{B}(\ell_2))$. Setting $y = \sum_{k \geq 0} x_k \otimes e_{k1}$, we conclude that

$$\left\| \sum_{k \geq 0} u_{k-1}(x_k) \otimes e_{k1} \right\|_{E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{B}(\ell_2))} \lesssim_E \left\| \sum_{k \geq 0} x_k \otimes e_{k1} \right\|_{E(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2))} = \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})},$$

which is our desired inequality and the proof is complete. \square

Proof of Theorem 1.5. It follows from Proposition 5.7 that

$$\begin{aligned} & \left\| \left(\sum_{k \geq 0} |x_k|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} |x_k^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M})} \\ & \geq \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k^*|^2) \right)^{1/2} \right\|_{E(\mathcal{M})}. \end{aligned}$$

Applying Theorem 1.3 to the above estimate, we obtain

$$(5.5) \quad \left\| \sum_k x_k \right\|_{E(\mathcal{M})} \gtrsim_E \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k^*|^2) \right)^{1/2} \right\|_{E(\mathcal{M})}.$$

By Theorem 1.1, we have

$$(5.6) \quad \left\| \sum_k x_k \right\|_E \gtrsim_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{Z_E^2(\mathcal{M} \bar{\otimes} \ell_\infty)}.$$

The assertion follows now by combining Proposition 5.5, (5.5) and (5.6). \square

As noted in the introduction, Theorem 1.5 partly resolves one question stated in [48]. At the time of writing this paper, we do not know how to extend Theorem 1.5 to the case that $E \in \text{Int}(L_2, L_q)$ for some $4 < q < \infty$. However, we have the following substitute, given in Theorem 5.10 below. We start with a technical lemma.

Lemma 5.8. *Let Φ be a 2-convex and q -concave Orlicz function such that*

$$(5.7) \quad \lim_{t \rightarrow 0} \frac{\Phi(t)}{t^2} = 0, \quad \lim_{t \rightarrow \infty} \frac{\Phi(t)}{t^q} = 0, \quad 2 < q < \infty.$$

Suppose that \mathcal{M} be a semifinite infinite von Neumann algebra and $x \in L_\Phi(\mathcal{M})$. Then there exists a measurable function x_Φ on $(0, \infty)$ such that

$$\|x\|_{L_\Phi(\mathcal{M})} \approx_q \|x \otimes x_\Phi\|_{(L_2 + L_q)(\mathcal{M} \otimes L_\infty(0, \infty))}.$$

Proof. Let Φ satisfy the assumptions. By [2, Lemma 6], the function ϕ defined by the setting $\Phi(t) = t^2 \phi(t^{q-2})$ is quasi-concave. By [35, Theorem II.1.1], there exists a concave increasing function ϕ_0 such that $\frac{1}{2} \phi_0 \leq \phi \leq \phi_0$. It follows from (5.7) that $\phi_0(0) = 0$ and $\lim_{t \rightarrow \infty} \frac{\phi_0(t)}{t} = 0$. By [11, Lemma 5.4.3], we have

$$\phi_0(t) = \int_0^\infty \min\{t, \tau\} d(-\phi'_0(\tau)).$$

Therefore,

$$\begin{aligned} \Phi(t) & \approx \int_0^\infty \min\{t^q, \tau t^2\} d(-\phi'_0(\tau)) \stackrel{\tau = s^{2-q}}{=} \int_0^\infty \min\{t^q, s^{2-q} t^2\} d(\phi'_0(s^{2-q})) \\ & = \int_0^\infty \min\{(ts)^q, (ts)^2\} d\nu(s) \approx_q \int_0^\infty N_q(ts) d\nu(s) \stackrel{\text{def}}{=} \Phi_0(t), \end{aligned}$$

where the positive measure $d\nu(s) = s^{-q}d(\phi'_0(s^{2-q}))$ and the Orlicz function N_q is defined by the setting

$$N_q(t) = \begin{cases} 2t^q, & 0 \leq t \leq 1 \\ qt^2 - q + 2, & t \geq 1. \end{cases}$$

The equivalence $\Phi(t) \approx_q \Phi_0(t)$ for all $t > 0$ guarantees that $L_\Phi(\mathcal{M}) = L_{\Phi_0}(\mathcal{M})$. Now let $\|x\|_{L_{\Phi_0}(\mathcal{M})} = \lambda$. It follows from $\mu(x \otimes y) = \mu(\mu(x) \otimes \mu(y))$ that

$$\begin{aligned} 1 &= \int_0^\infty \Phi_0\left(\frac{\mu(t, x)}{\lambda}\right) dt = \int_0^\infty \int_0^\infty N_q\left(\frac{\mu(t, x)s}{\lambda}\right) d\nu(s) dt \\ &= (\tau \otimes \int) \left(N_q\left(\frac{x \otimes x_\Phi}{\lambda}\right) \right), \end{aligned}$$

where x_Φ is a measurable function on $(0, \infty)$ whose distribution is ν . Now the well-known property of Orlicz norms (see e.g. [45, I,1.2, Proposition 11]) implies that

$$\|x\|_{L_\Phi(\mathcal{M})} \approx_q \|x\|_{L_{\Phi_0}(\mathcal{M})} = \|x \otimes x_\Phi\|_{L_{N_q}(\mathcal{M} \otimes L_\infty(0, \infty))}.$$

One can easily show that $(L_2 + L_q)(0, \infty) = L_{N_q}(0, \infty)$, and we conclude this proof. \square

The following assertion is proved in [33, Theorem 7.1] (see also [5, Theorem 2.7]).

Theorem 5.9. *Let $E \in \text{Int}(L_p, L_q)$, $1 \leq p < q \leq \infty$, be a symmetric Banach function space. If z and w are such that $\|z\|_{L_\Phi} \leq \|w\|_{L_\Phi}$ for every p -convex and q -concave Orlicz function Φ and $w \in E$, then $z \in E$ and $\|z\|_E \lesssim_E \|w\|_E$.*

The following result shows that the remaining part concerning the Burkholder inequality for symmetric Banach spaces $E \in \text{Int}(L_2, L_q)$ for $4 < q < \infty$ needs to be answered only for $E = L_2 + L_q$.

Theorem 5.10. *Let \mathcal{M} be a semifinite infinite von Neumann algebra. Suppose that for every sequence $(x_k)_{k \geq 0} \subset (L_2 + L_q)(\mathcal{M})$ of martingale differences, $2 < q < \infty$, the inequality*

$$\begin{aligned} \left\| \sum_{k \geq 0} x_k \right\|_{(L_2 + L_q)(\mathcal{M})} &\lesssim_q \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{(L_2 + L_q)(\mathcal{M} \bar{\otimes} \ell_\infty)} \\ &+ \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{(L_2 + L_q)(\mathcal{M})} \\ &+ \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k^*|^2) \right)^{1/2} \right\|_{(L_2 + L_q)(\mathcal{M})} \end{aligned}$$

holds. If $E \in \text{Int}(L_2, L_q)$ is a symmetric Banach function space on the semi-axis, then for every sequence $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ of martingale differences the inequality

$$\begin{aligned} \left\| \sum_{k \geq 0} x_k \right\|_{E(\mathcal{M})} &\lesssim_E \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{E(\mathcal{M} \bar{\otimes} \ell_\infty)} + \\ &+ \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} + \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k^*|^2) \right)^{1/2} \right\|_{E(\mathcal{M})} \end{aligned}$$

holds.

Proof. Let $(x_k)_{k \geq 0} \subset E(\mathcal{M})$ be an arbitrary sequence of martingale differences. Let Φ be as in Lemma 5.8. Take x_Φ on $(0, \infty)$ from Lemma 5.8. Consider a sequence $(x_k \otimes x_\Phi)_{k \geq 0}$. This sequence consists of martingale differences with respect to the filtration $(\mathcal{M}_k \otimes L_\infty(0, \infty))_{k \geq 0}$. By Lemma 5.8 we have

$$(5.8) \quad \left\| \sum_{k \geq 0} x_k \right\|_{L_\Phi(\mathcal{M})} \approx_q \left\| \sum_{k \geq 0} x_k \otimes x_\Phi \right\|_{(L_2+L_q)(\mathcal{M} \bar{\otimes} L_\infty(0, \infty))}.$$

Applying the assumption to the sequence $(x_k \otimes x_\Phi)_{k \geq 0}$ and combining (5.8) we obtain

$$\begin{aligned} \left\| \sum_{k \geq 0} x_k \right\|_{L_\Phi(\mathcal{M})} &\lesssim_q \left\| \sum_{k \geq 0} (x_k \otimes x_\Phi) \otimes e_k \right\|_{(L_2+L_q)(\mathcal{M} \bar{\otimes} L_\infty(0, \infty) \bar{\otimes} \ell_\infty)} \\ &+ \left\| \left(\sum_{k \geq 0} (\mathcal{E}_{k-1} \otimes 1)(|x_k|^2 \otimes x_\Phi^2) \right)^{1/2} \right\|_{(L_2+L_q)(\mathcal{M} \bar{\otimes} L_\infty(0, \infty))} \\ &+ \left\| \left(\sum_{k \geq 0} (\mathcal{E}_{k-1} \otimes 1)(|x_k^*|^2 \otimes x_\Phi^2) \right)^{1/2} \right\|_{(L_2+L_q)(\mathcal{M} \bar{\otimes} L_\infty(0, \infty))} \\ &= \left\| \left(\sum_{k \geq 0} x_k \otimes e_k \right) \otimes x_\Phi \right\|_{(L_2+L_q)(\mathcal{M} \bar{\otimes} \ell_\infty \bar{\otimes} L_\infty(0, \infty))} \\ &+ \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \otimes x_\Phi \right\|_{(L_2+L_q)(\mathcal{M} \bar{\otimes} L_\infty(0, \infty))} \\ &+ \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k^*|^2) \right)^{1/2} \otimes x_\Phi \right\|_{(L_2+L_q)(\mathcal{M} \bar{\otimes} L_\infty(0, \infty))} \\ &\approx_q \left\| \sum_{k \geq 0} x_k \otimes e_k \right\|_{L_\Phi(\mathcal{M} \bar{\otimes} \ell_\infty)} + \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k|^2) \right)^{1/2} \right\|_{L_\Phi(\mathcal{M})} \\ &+ \left\| \left(\sum_{k \geq 0} \mathcal{E}_{k-1}(|x_k^*|^2) \right)^{1/2} \right\|_{L_\Phi(\mathcal{M})}. \end{aligned}$$

The assertion follows now from Theorem 5.9. \square

6. DISJOINTIFICATION INEQUALITIES: THE Φ -MOMENT CASE

In this section, we prove Φ -moment version of **disjointification** inequalities for noncommutative martingale difference sequences. The main result is Theorem 1.6, which is new even in the classical case.

We consider the Orlicz function

$$(6.1) \quad M(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ 2t - 1, & t \geq 1. \end{cases}$$

Lemma 6.1. *Let (\mathcal{M}, τ) and (\mathcal{N}, σ) be finite von Neumann algebras. If $T : L_1(\mathcal{N}) \rightarrow L_1(\mathcal{M})$ and $T : L_2(\mathcal{N}) \rightarrow L_2(\mathcal{M})$ is a contraction, then*

$$\tau(M(|Tx|)) \leq 4\sigma(M(|x|)).$$

Proof. Split $x = x_1 + x_2$, where $x_1 = xe_{[0,1]}(|x|)$ and $x_2 = xe_{(1,\infty)}(|x|)$. By [38, Theorem 3.3.3], we have

$$\mu(Tx) = \mu(Tx_1 + Tx_2) \prec\prec \mu(Tx_1) + \mu(Tx_2) = \frac{1}{2}(\mu(2Tx_1) + \mu(2Tx_2)).$$

and, therefore,

$$\tau(M(|Tx|)) \leq \mathbb{E}\left(M\left(\frac{1}{2}(\mu(2Tx_1) + \mu(2Tx_2))\right)\right) \leq \frac{1}{2}\left(\tau(M(2|Tx_1|)) + \tau(M(2|Tx_2|))\right).$$

It is easy to see $M(2t) \leq 4M(t)$, $M(t) \leq 2t$ and $M(t) \leq t^2$ for every $t > 0$, and hence we get

$$\tau(M(|Tx|)) \leq 2\tau(M(|Tx_1|)) + 2\tau(M(|Tx_2|)) \leq 2\tau(|Tx_1|^2) + 4\tau(|Tx_2|).$$

Since $T : L_1(\mathcal{N}) \rightarrow L_1(\mathcal{M})$ and $T : L_2(\mathcal{N}) \rightarrow L_2(\mathcal{M})$ is a contraction, it follows that

$$\tau(M(|Tx|)) \leq 2\sigma(|x_1|^2) + 4\sigma(|x_2|) \leq 4\sigma(M(|x_1|)) + 4\sigma(M(|x_2|)) = 4\sigma(M(|x|)).$$

□

Lemma 6.2. *Let (\mathcal{M}, τ) be a finite von Neumann algebra and let (\mathcal{N}, ν) be an atomless semifinite one. If $T : L_2(\mathcal{N}) \rightarrow L_2(\mathcal{M})$ and $T : L_1(\mathcal{N}) \rightarrow L_1(\mathcal{M})$ is a linear contraction, then for M given by (6.1) we have*

$$\tau(M(|Tx|)) \lesssim \left(\int_0^1 M(\mu(s, x))ds + M(\|x\|_{L_1+L_2})\right).$$

Proof. To see this, fix $x = x^* \in L_0(\mathcal{N})$ and consider the spectral projection

$$p = e_{(\mu(1, x), \infty)}(|x|).$$

Clearly, trace of p is less than or equal to 1. Fix a projection $q \geq p$ whose trace equals 1. Clearly, $T : L_1(q\mathcal{N}q) \rightarrow L_1(\mathcal{M})$ and $T : L_2(q\mathcal{N}q) \rightarrow L_2(\mathcal{M})$ is a contraction. Again by [38, Theorem 3.3.3], we have

$$\mu(Tx) \prec\prec \mu(T(pxp)) + \mu(T((1-p)xp)) + \mu(T(x(1-p))).$$

Therefore,

$$\begin{aligned} \tau(M(|Tx|)) &\leq \mathbb{E}\left(M\left(\mu(T(pxp)) + \mu(T((1-p)xp)) + \mu(T(x(1-p)))\right)\right) \\ &\lesssim \tau(M(T(3|pxp|))) + \tau(M(3|T((1-p)xp)|)) + \tau(M(3|T(x(1-p))|)). \end{aligned}$$

Applying Lemma 6.1 to the finite algebras \mathcal{M} and $q\mathcal{N}q$ and noting that $M(3t) \leq 9M(t)$ for every $t > 0$, we obtain that

$$\tau(M(T(3|pxp|))) \leq 4\sigma(M(3|pxp|)) \leq 36\sigma(M(|pxp|)) \leq 36 \int_0^1 M(\mu(s, x))ds.$$

Define concave function M_0 by setting $M_0(t) = M(t^{1/2})$, $t > 0$. It follows from Jensen inequality and the assumption that

$$\tau(M(|Tz|)) = \tau(M_0(|Tz|^2)) \leq M_0(\tau(|Tz|^2)) = M_0(\|Tz\|_2^2) = M(\|Tz\|_2) \leq M(\|z\|_2).$$

Therefore,

$$\begin{aligned} \tau(M(3|T((1-p)xp)|)) + \tau(M(3|T(x(1-p))|)) &\leq 2M(3\|x(1-p)\|_2) \\ &= 2M(3\|\mu(x)\chi_{(1, \infty)}\|_2) \lesssim 2M(3\|x\|_{L_1+L_2}) \leq 18M(\|x\|_{L_1+L_2}). \end{aligned}$$

□

Lemma 6.3. *Let (\mathcal{M}, τ) be a finite von Neumann algebra and let (\mathcal{N}, ν) be a semifinite atomless one. Let Φ be a 2-concave Orlicz function. If $T : L_2(\mathcal{N}) \rightarrow L_2(\mathcal{M})$ and $T : L_1(\mathcal{N}) \rightarrow L_1(\mathcal{M})$ is a linear contraction, then*

$$\tau(\Phi(|Tx|)) \lesssim \int_0^1 \Phi(\mu(s, x)) ds + \Phi(\|x\|_{L_1+L_2}).$$

Proof. Define the function $\phi : (0, \infty) \rightarrow (0, \infty)$ by setting $\Phi(t) = t\phi(t)$, $t > 0$. Since Φ is convex and 2-concave, it follows that ϕ is quasi-concave; see [2, Lemma 6]. By [35, Theorem II.1.1], there exists a concave increasing function ϕ_0 such that $\frac{1}{2}\phi_0 \leq \phi \leq \phi_0$. Without loss of generality, $\phi_0(0) = 0$ and $\phi_0(t) = o(t)$, $t \rightarrow \infty$. By [11, Lemma 5.4.3], we have

$$\phi_0(t) = \int_0^\infty \min\{t, \lambda\} d(-\phi'_0(\lambda)).$$

Arguing as in Lemma 5.8, we obtain that

$$\Phi(t) \approx \int_0^\infty M\left(\frac{t}{\lambda}\right) d\nu(\lambda)$$

for some (positive) measure ν on $(0, \infty)$. Hence,

$$\tau(\Phi(|x|)) \approx \int_0^\infty \tau\left(M\left(\frac{|x|}{\lambda}\right)\right) d\nu(\lambda).$$

Therefore, by Lemma 6.2, we have

$$\begin{aligned} \tau(\Phi(|Tx|)) &\approx \int_0^\infty \tau\left(M\left(\frac{|Tx|}{\lambda}\right)\right) d\nu(\lambda) \\ &\lesssim \int_0^\infty \left(\int_0^1 M\left(\mu(s, \frac{x}{\lambda})\right) ds + M\left(\frac{1}{\lambda} \|x\|_{L_1+L_2}\right) \right) d\nu(\lambda) \\ &= \int_0^1 \int_0^\infty M\left(\mu(s, \frac{x}{\lambda})\right) d\nu(\lambda) ds + \int_0^\infty M\left(\frac{1}{\lambda} \|x\|_{L_1+L_2}\right) d\nu(\lambda) \\ &\approx \int_0^1 \Phi(\mu(s, x)) ds + \Phi(\|x\|_{L_1+L_2}), \end{aligned}$$

which is as our desired inequality and the proof is complete. \square

Lemma 6.4. *Let Φ be a 2-convex Orlicz function. We have*

$$(\tau \otimes \int) \left(\Phi \left(\left| \sum_{k \geq 0} y_k \otimes r_k \right| \right) \right) \geq \tau(\Phi((\sum_{k \geq 0} |y_k|^2)^{1/2})).$$

Proof. Without loss of generality, we have finitely many (let us say, n) summands in the inequality above. Let \mathcal{R}_n be the set of all maps from $\{0, \dots, n-1\}$ into $\{-1, 1\}$, that is

$$\mathcal{R}_n = \left\{ \epsilon : \{0, \dots, n-1\} \rightarrow \{-1, 1\} \right\}.$$

For every $\epsilon \in \mathcal{R}_n$, we set

$$T_\epsilon := \left| \sum_{k=0}^{n-1} \epsilon(k) y_k \right|^2 \quad \text{and} \quad A_\epsilon := \{t \in (0, 1) : r_k(t) = \epsilon(k), 0 \leq k \leq n-1\}.$$

It is clear that the Lebesgue measure of A_ϵ is 2^{-n} . Set $\Phi_0(t) = \Phi(t^{1/2})$, $t > 0$. Since Φ is 2-convex, it follows that Φ_0 is convex. Hence, we have

$$(6.2) \quad (\tau \otimes \int) \left(\Phi \left(\left| \sum_{k=0}^{n-1} y_k \otimes r_k \right| \right) \right) = \sum_{\epsilon \in \mathcal{R}_n} (\tau \otimes \int_{A_\epsilon}) \left(\Phi \left(\left| \sum_{k=0}^{n-1} y_k \otimes r_k \right| \right) \right) = 2^{-n} \sum_{\epsilon \in \mathcal{R}_n} \tau(\Phi_0(T_\epsilon)).$$

Applying (2.3), we obtain

$$(6.3) \quad 2^{-n} \sum_{\epsilon \in \mathcal{R}_n} \tau(\Phi_0(T_\epsilon)) \geq \tau(\Phi_0(2^{-n} \sum_{\epsilon \in \mathcal{R}_n} T_\epsilon)).$$

Note that for $0 \leq l, m \leq n-1$,

$$\sum_{\epsilon \in \mathcal{R}_n} \epsilon(l)\epsilon(m) = \begin{cases} 2^n, & l = m \\ 0, & l \neq m. \end{cases}$$

It is immediate that

$$2^{-n} \sum_{\epsilon \in \mathcal{R}_n} T_\epsilon = 2^{-n} \sum_{\epsilon \in \mathcal{R}_n} \left| \sum_{k=0}^{n-1} \epsilon(k) y_k \right|^2 = 2^{-n} \sum_{l,m=0}^{n-1} y_l^* y_m \sum_{\epsilon \in \mathcal{R}_n} \epsilon(l)\epsilon(m) = \sum_{k=0}^{n-1} |y_k|^2.$$

Consequently,

$$(6.4) \quad \tau(\Phi_0(2^{-n} \sum_{\epsilon \in \mathcal{R}_n} T_\epsilon)) = \tau(\Phi_0(\sum_{k=0}^{n-1} |y_k|^2)) = \tau(\Phi((\sum_{k=0}^{n-1} |y_k|^2)^{1/2})).$$

Combining (6.2), (6.3) and (6.4), we conclude the proof. \square

Lemma 6.5. *Let Φ be an arbitrary Orlicz function. If $0 \leq z_k \in L_\Phi(\mathcal{M})$, $k \geq 0$, then we have*

$$\tau(\Phi(\sum_{k \geq 0} z_k)) \geq \frac{1}{2} \left(\mathbb{E}(\Phi(\mu(Z)\chi_{(0,1)})) + \Phi(\|Z\|_1) \right), \quad Z = \sum_{k \geq 0} z_k \otimes e_k.$$

Proof. Without loss of generality, suppose that the spectral projection $e_t(z_k) = 0$ for every $t > 0$. Set

$$z_{1k} = z_k e_{(\mu(1,Z), \infty)}(z_k), \quad z_{2k} = z_k e_{[0, \mu(1,Z)]}(z_k).$$

By [38, Lemma 3.3.7], we have

$$\sum_{k \geq 0} z_{1k} \otimes e_k \prec\prec \sum_{k \geq 0} z_{1k}$$

and, therefore,

$$\mu(Z)\chi_{(0,1)} \prec\prec \mu\left(\sum_{k \geq 0} z_k\right).$$

Thus, by (2.5)

$$\tau(\Phi(\sum_{k \geq 0} z_k)) \geq \mathbb{E}(\Phi(\mu(Z)\chi_{(0,1)})).$$

By Jensen inequality, we have

$$\tau(\Phi(\sum_{k \geq 0} z_k)) = \int_0^1 \Phi(\mu(t, \sum_{k \geq 0} z_k)) dt \geq \Phi(\tau(\sum_{k \geq 0} z_k)) = \Phi(\sum_{k \geq 0} \|z_k\|_1) = \Phi(\|Z\|_1).$$

The assertion follows by combining the latter two inequalities. \square

Now we are in a position to prove our third main result, Theorem 1.6.

Proof of Theorem 1.6. We first prove the first assertion. Consider the operator T defined as the proof of Proposition 3.2. Then T is both bounded from $L_1(\mathcal{M} \bar{\otimes} \ell_\infty)$ to $L_1(\mathcal{M})$ and from $L_2(\mathcal{M} \bar{\otimes} \ell_\infty)$ to $L_2(\mathcal{M})$. Suppose now $\{x_k\}_{k \geq 0}$ is a sequence of martingale differences. Applying Lemma 6.3, we have

$$\tau(\Phi(|\sum_{k \geq 0} x_k|)) = \tau(\Phi(|TX|)) \lesssim \int_0^1 \Phi(\mu(s, X)) ds + \Phi(\|X\|_{L_1+L_2}).$$

We now turn to the proof of the second assertion. Since Φ is 2-convex and q -concave, it follows from [9, Corollary 3.1] that, for every $t \in (0, 1)$

$$\tau\left(\Phi\left(|\sum_{k \geq 0} x_k|\right)\right) \approx_\Phi \tau\left(\Phi\left(|\sum_{k \geq 0} r_k(t)x_k|\right)\right).$$

Hence,

$$\tau\left(\Phi\left(|\sum_{k \geq 0} x_k|\right)\right) \approx_\Phi (\tau \otimes \int)\left(\Phi\left(|\sum_{k \geq 0} r_k \otimes x_k|\right)\right).$$

Set $\Phi_0(t) := \Phi(t^{1/2})$, $t > 0$. It follows from Lemma 6.4 that

$$(6.5) \quad \tau\left(\Phi\left(|\sum_{k \geq 0} x_k|\right)\right) \gtrsim_\Phi \tau\left(\Phi\left((\sum_{k \geq 0} |x_k|^2)^{1/2}\right)\right) = \tau\left(\Phi_0\left(\sum_{k \geq 0} |x_k|^2\right)\right).$$

Since Φ is 2-convex, it follows that Φ_0 is convex. By Lemma 6.5, we have

$$\begin{aligned} \tau\left(\Phi_0\left(\sum_{k \geq 0} |x_k|^2\right)\right) &\geq \frac{1}{2} \left(\mathbb{E}(\Phi_0(\mu(\sum_{k \geq 0} |x_k|^2 \otimes e_k) \chi_{(0,1)})) + \Phi_0(\|\sum_{k \geq 0} |x_k|^2 \otimes e_k\|_1) \right) \\ &\geq \frac{1}{2} \left(\mathbb{E}(\Phi_0(\mu^2(X) \chi_{(0,1)})) + \Phi_0(\|X\|_2^2) \right) \\ &= \frac{1}{2} \mathbb{E}(\Phi(\mu(X) \chi_{(0,1)})) + \Phi(\|X\|_2). \end{aligned}$$

Combining this inequality and (6.5), we conclude the proof. \square

Now we show that the assumption on Φ in Theorem 1.6 (ii) is sharp in the sense that the condition of q -concavity is necessary.

Proposition 6.6. *Let (\mathcal{M}, τ) be an arbitrary noncommutative probability space. Suppose that an Orlicz function Φ is such that for every sequence of martingale differences $(x_k)_{k \geq 0} \subset L_\Phi(\mathcal{M})$ we have*

$$\mathbb{E}\left(\Phi(\mu(X) \chi_{(0,1)})\right) \lesssim_\Phi \tau\left(\Phi\left(\sum_{k \geq 0} x_k\right)\right), \quad X = \sum_{k \geq 0} x_k \otimes e_k.$$

It follows that Φ is q -concave for some $q < \infty$.

Proof. Let $\mathcal{M}_0 = \mathbb{C}$ and take $x \in \mathcal{M}_1$. That is, our sequence $\{x_k\}_{k \geq 0}$ is defined by the setting $x_0 = \tau(x)$, $x_1 = x - \tau(x)$ and $x_k = 0$, $k > 1$. **Let $t \in \mathbb{R}_+$ be an arbitrary fixed constant. Take**

$$x = 9t\chi_{(0, \frac{1}{3})} - 6t\chi_{(\frac{1}{3}, 1)}.$$

We have

$$x_0 = -t, \quad x_1 = 10t\chi_{(0, \frac{1}{3})} - 5t\chi_{(\frac{1}{3}, 1)}.$$

Hence,

$$\mu(X) = 10t\chi_{(0, \frac{1}{3})} + 5t\chi_{(\frac{1}{3}, 1)} + t\chi_{(1, 2)}.$$

Therefore, by the assumption, we have

$$\frac{1}{3}\Phi(10t) + \frac{2}{3}\Phi(5t) \lesssim_{\Phi} \frac{1}{3}\Phi(9t) + \frac{2}{3}\Phi(6t).$$

Hence,

$$\Phi(10t) \lesssim_{\Phi} \Phi(9t).$$

This implies that Φ satisfies Δ_2 -condition, and hence Φ is q -concave. Indeed, if Φ satisfies Δ_2 -condition, then it is easy to find $0 < q < \infty$ such that $\frac{\Phi(t)}{t^q}, t > 0$ is decreasing. Since Φ is convex, $\frac{\Phi(t)}{t}, t > 0$ is increasing. By [31, Lemma 8], we have that Φ is q -concave. \square

7. NONCOMMUTATIVE MARTINGALE INEQUALITIES: THE Φ -MOMENT CASE

In this section we prove the Φ -moment Burkholder-Gundy inequalities by using results established in the preceding section.

Lemma 7.1. *Let Φ be an Orlicz function and let $(x_k)_{k \geq 0}$ be an arbitrary sequence from $L_{\Phi}(\mathcal{M})$. If Φ is an 2-concave Orlicz function, then*

$$(\tau \otimes \int) \left(\Phi \left(\left| \sum_{k \geq 0} x_k \otimes r_k \right| \right) \right) \lesssim \tau(\Phi((\sum_{k \geq 0} |x_k|^2)^{1/2})).$$

Proof. Consider the operator S defined in Lemma 4.1. Observe that the operator S satisfies the assumptions in Lemma 6.3. Applying this lemma, we have

$$(\tau \otimes \int) (\Phi(|Sx|)) \lesssim \int_0^1 \Phi(\mu(s, x)) ds + \Phi(\|x\|_{L_1+L_2}), \quad x \in L_0(\mathcal{M} \bar{\otimes} B(\ell_2)).$$

Setting $x = \sum_{k \geq 0} x_k \otimes e_{k1}$, we write

$$(\tau \otimes \int) \left(\Phi \left(\left| \sum_{k \geq 0} x_k \otimes r_k \right| \right) \right) \lesssim \int_0^1 \Phi(\mu(s, x)) ds + \Phi(\|x\|_{L_1+L_2}).$$

However,

$$\mu\left(s, \sum_{k \geq 0} x_k \otimes e_{k1}\right) = \mu\left(s, \left(\sum_{k \geq 0} |x_k|^2\right)^{1/2}\right), \quad s > 0$$

which lives on $(0, 1)$, and therefore,

$$\begin{aligned} (\tau \otimes \int) \left(\Phi \left(\left| \sum_{k \geq 0} x_k \otimes r_k \right| \right) \right) &\lesssim \tau(\Phi((\sum_{k \geq 0} |x_k|^2)^{1/2})) + \Phi\left(\int_0^1 \mu\left(s, \left(\sum_{k \geq 0} |x_k|^2\right)^{1/2}\right) ds\right) \\ &\leq \tau(\Phi((\sum_{k \geq 0} |x_k|^2)^{1/2})) + \int_0^1 \Phi\left(\mu\left(s, \left(\sum_{k \geq 0} |x_k|^2\right)^{1/2}\right)\right) ds \\ &= 2\tau(\Phi((\sum_{k \geq 0} |x_k|^2)^{1/2})). \end{aligned}$$

This proof is complete. \square

The following result is the main result of this section. It should be compared with [9, Theorem 5.1].

Theorem 7.2. *Let Φ be an Orlicz function and let $(x_k)_{k \geq 0} \in L_{\Phi}(\mathcal{M})$ be a sequence of martingale differences.*

(i) If Φ is p -convex for some $1 < p < 2$ and 2-concave, then

$$\tau\left(\Phi\left(\left|\sum_{k \geq 0} x_k\right|\right)\right) \approx_{\Phi} \inf_{x_k = y_k + z_k} \left\{ \tau\left(\Phi\left[\left(\sum_{k \geq 0} |y_k|^2\right)^{1/2}\right]\right) + \tau\left(\Phi\left[\left(\sum_{k \geq 0} |z_k^*|^2\right)^{1/2}\right]\right) \right\},$$

where the infimum is taken over the sequences of martingale differences.

(ii) If Φ is 2-convex and q -concave for some $2 < q < \infty$, then

$$\tau\left(\Phi\left(\left|\sum_{k \geq 0} x_k\right|\right)\right) \approx_{\Phi} \max \left\{ \tau\left(\Phi\left[\left(\sum_{k \geq 0} |x_k|^2\right)^{1/2}\right]\right), \tau\left(\Phi\left[\left(\sum_{k \geq 0} |x_k^*|^2\right)^{1/2}\right]\right) \right\}.$$

Proof. Since Φ is p -convex and q -concave for $1 < p < q < \infty$, it follows from [9, Corollary 3.1] that

$$(7.1) \quad \tau\left(\Phi\left(\left|\sum_{k \geq 0} x_k\right|\right)\right) \approx_{\Phi} (\tau \otimes \int)\left(\Phi\left(\left|\sum_{k \geq 0} x_k \otimes r_k\right|\right)\right).$$

(i) It follows from (7.1) and the proof of [9, Theorem 4.1 (i)]⁵ that

$$\tau\left(\Phi\left(\left|\sum_{k \geq 0} x_k\right|\right)\right) \gtrsim_{\Phi} \inf_{x_k = y_k + z_k} \left\{ \tau\left(\Phi\left[\left(\sum_{k \geq 0} |y_k|^2\right)^{1/2}\right]\right) + \tau\left(\Phi\left[\left(\sum_{k \geq 0} |z_k^*|^2\right)^{1/2}\right]\right) \right\},$$

where the infimum is taken over all possible decompositions $x_k = y_k + z_k$ with y_k and z_k in $L_{\Phi}(\mathcal{M})$. For any fixed decomposition $x_k = y_k + z_k$, since $(x_k)_{k \geq 0}$ is a martingale difference sequence,

$$x_k = (\mathcal{E}_k(y_k) - \mathcal{E}_{k-1}(y_k)) + (\mathcal{E}_k(y_k) - \mathcal{E}_{k-1}(y_k)) := u_k + v_k.$$

Then $(u_k)_{k \geq 0}$ and $(v_k)_{k \geq 0}$ are martingale difference sequences, and by the Stein inequality [9, Theorem 3.2] and (2.4), we have

$$\tau\left(\Phi\left[\left(\sum_{k \geq 0} |u_k|^2\right)^{1/2}\right]\right) \lesssim_{\Phi} \tau\left(\Phi\left[\left(\sum_{k \geq 0} |y_k|^2\right)^{1/2}\right]\right)$$

and

$$\tau\left(\Phi\left[\left(\sum_{k \geq 0} |v_k^*|^2\right)^{1/2}\right]\right) \lesssim_{\Phi} \tau\left(\Phi\left[\left(\sum_{k \geq 0} |z_k^*|^2\right)^{1/2}\right]\right).$$

Hence, we have

$$\tau\left(\Phi\left(\left|\sum_{k \geq 0} x_k\right|\right)\right) \gtrsim_{\Phi} \inf_{x_k = y_k + z_k} \left\{ \tau\left(\Phi\left[\left(\sum_{k \geq 0} |y_k|^2\right)^{1/2}\right]\right) + \tau\left(\Phi\left[\left(\sum_{k \geq 0} |z_k^*|^2\right)^{1/2}\right]\right) \right\},$$

where the infimum is taken over the sequences of martingale differences.

We now prove the converse inequality. Let $x_k = y_k + z_k$, with $(y_k)_{k \geq 0} \subset E(\mathcal{M})$ and $(z_k)_{k \geq 0} \subset E(\mathcal{M})$ being sequences of martingale differences. Applying Lemma 7.1 and (7.1), we conclude that

$$\tau\left(\Phi\left(\left|\sum_{k \geq 0} y_k\right|\right)\right) \lesssim_{\Phi} \tau\left(\Phi\left[\left(\sum_{k \geq 0} |y_k|^2\right)^{1/2}\right]\right)$$

and, similarly,

$$\tau\left(\Phi\left(\left|\sum_{k \geq 0} z_k\right|\right)\right) \lesssim_{\Phi} \tau\left(\Phi\left[\left(\sum_{k \geq 0} |z_k^*|^2\right)^{1/2}\right]\right).$$

⁵Strictly speaking, the argument of [9, Theorem 4.1 (i)] works well for every p -convex and q -concave Orlicz function with $1 < p < q < \infty$ without any change.

Therefore, again by (2.4) we have

$$\tau\left(\Phi\left(\left|\sum_{k \geq 0} x_k\right|\right)\right) \lesssim_{\Phi} \inf_{x_k = y_k + z_k} \left\{ \tau\left(\Phi\left[\left(\sum_{k \geq 0} |y_k|^2\right)^{1/2}\right]\right) + \tau\left(\Phi\left[\left(\sum_{k \geq 0} |z_k^*|^2\right)^{1/2}\right]\right) \right\},$$

where the infimum is taken over the sequences of martingale differences.

(ii) The inequality

$$\tau\left(\Phi\left(\left|\sum_{k \geq 0} x_k\right|\right)\right) \lesssim_{\Phi} \max \left\{ \tau\left(\Phi\left[\left(\sum_{k \geq 0} |x_k|^2\right)^{1/2}\right]\right), \tau\left(\Phi\left[\left(\sum_{k \geq 0} |x_k^*|^2\right)^{1/2}\right]\right) \right\},$$

is established in [18, Corollary 3.3]. The inequality

$$\tau\left(\Phi\left(\left|\sum_{k \geq 0} x_k\right|\right)\right) \gtrsim_{\Phi} \max \left\{ \tau\left(\Phi\left[\left(\sum_{k \geq 0} |x_k|^2\right)^{1/2}\right]\right), \tau\left(\Phi\left[\left(\sum_{k \geq 0} |x_k^*|^2\right)^{1/2}\right]\right) \right\}.$$

follows from Lemma 6.4 and (7.1). \square

We end this paper by giving some examples to illustrate that the Burkholder-Gundy inequalities in Theorem 1.3 (respectively, Theorem 7.2) can be applied to a wider class of symmetric spaces (respectively, Orlicz functions) than the main results in [17, 24] (respectively, [9]).

Example 7.3. Consider the family of functions

$$\Phi(t) = t^p \ln^{\alpha}(e + t), \quad t \geq 0,$$

where $1 < p < \infty$ and $\alpha \in \mathbb{R}$. The function Φ is a convex function for $\alpha \geq 0$ and is equivalent to a convex function for $\alpha < 0$. It is easy to see that $p_{\Phi} = q_{\Phi} = p$. Moreover, Φ is p -convex for $\alpha \geq 0$ and r -convex for each $1 \leq r < p$ if $\alpha < 0$ [50, page 284]. If $p = 2$, then [9, Theorem 5.1] is not applicable, whereas Theorem 7.2 above still yields Φ -moment Burkholder-Gundy inequalities.

Example 7.4. Let $\Phi(t) = t^p \log(1 + t^q)$ with $p > 1$ and $q > 0$. It is easy to check that Φ is an Orlicz function with $p_{\Phi} = p$ and $q_{\Phi} = p + q$.

- (i) Suppose that $p = 2$. Then [9, Theorem 5.1] gives no information. However, by [31, Lemma 8] it is not hard to see that Φ is 2-convex and $(2+q)$ -concave, and hence the corresponding Burkholder-Gundy inequality holds due to Theorem 7.2 (ii).
- (ii) Suppose that $p + q = 2$. Then [9, Theorem 5.1] also gives no information. However, Φ is 2-concave, and hence the corresponding Burkholder-Gundy inequality holds due to Theorem 7.2 (i).

Example 7.5. Let Φ be as in Example 7.3 and Example 7.4 and let $L_{\Phi}(0, 1)$ be the corresponding Orlicz space. Then Theorem 1.3 can be applied to the symmetric space $E = L_{\Phi}(0, 1)$, while [17, Proposition 4.18] and [24, Theorem 3.1] are both inefficient. In particular, in the setting of Example 7.4 with $p = 2$ and $p + q \leq 4$, Theorem 1.5 is applicable to the symmetric space $E = L_{\Phi}(0, 1)$ and yields the corresponding Burkholder inequalities; in fact since Φ is 2-convex and $p+q$ -concave, $L_{\Phi}(0, 1)$ is 2-convex and $p+q$ -concave ([31, Theorem 50]), and hence $L_{\Phi}(0, 1) \in \text{Int}(L_2(0, 1), L_4(0, 1))$ ([33] or [15, Theorem 4.31]).

APPENDIX A. PROOF OF THEOREM 5.6

The proof below is due to Junge, Proposition 2.8 in [29].

We use the theory of Hilbert modules presented in [36] (in particular, Kasparov stabilisation theorem).

Hilbert modules $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ are said to be isomorphic (see p.24 in [36]) if there exists a linear bijection $w : H_1 \rightarrow H_2$ such that

$$\langle w(x), w(y) \rangle_2 = \langle x, y \rangle_1, \quad x, y \in H_1.$$

Consider the standard Hilbert module $H_{\mathcal{N}}$ as in [29],

$$H_{\mathcal{N}} := \{x \in \mathcal{N} \bar{\otimes} \mathcal{B}(\ell_2) : x \cdot (1 \otimes e_{11}) = x\},$$

and

$$x^*y = \langle x, y \rangle_{H_{\mathcal{N}}} \otimes e_{11}, \quad x, y \in H_{\mathcal{N}}.$$

Proof of Theorem 5.6. Let $X \subset \mathcal{M}$ be a separable C^* -subalgebra which is dense in $L_2(\mathcal{M})$. Let F be a closure of X with respect to the norm $x \rightarrow \|\mathcal{E}(|x|^2)\|_{\mathcal{N}}^{1/2}$. It follows that F is a countably generated Hilbert module over \mathcal{N} when equipped with the \mathcal{N} -valued inner product

$$(x, y) \rightarrow \mathcal{E}(x^*y).$$

By [36, Theorem 6.2], Hilbert modules $F \oplus H_{\mathcal{N}}$ and $H_{\mathcal{N}}$ are isomorphic. Hence, there exists a mapping $w : F \oplus H_{\mathcal{N}} \rightarrow H_{\mathcal{N}}$ such that

$$\langle w(x_1, x_2), w(y_1, y_2) \rangle_{H_{\mathcal{N}}} = \mathcal{E}(x_1^*y_1) + \langle x_2, y_2 \rangle_{H_{\mathcal{N}}}.$$

Setting $x_2 = y_2 = 0$, we obtain

$$\langle w(x_1, 0), w(y_1, 0) \rangle_{H_{\mathcal{N}}} = \mathcal{E}(x_1^*y_1).$$

Therefore,

$$w(x_1, 0)^*w(y_1, 0) = \mathcal{E}(x_1^*y_1) \otimes e_{11}.$$

Consider the mapping $u : F \rightarrow H_{\mathcal{N}}$ defined by the setting $u(x) = w(x, 0)$. In particular, $u : X \rightarrow \mathcal{N} \bar{\otimes} \mathcal{B}(\ell_2)$ and

$$u(x)^*u(y) = \mathcal{E}(x^*y) \otimes e_{11}, \quad x, y \in X.$$

Clearly,

$$\|u(x)\|_{L_2(\mathcal{N} \bar{\otimes} \mathcal{B}(\ell_2))} = \|(\mathcal{E}(|x|^2))^{1/2}\|_{L_2(\mathcal{N})} = \|x\|_{L_2(\mathcal{N})}.$$

Hence, $u : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{N} \bar{\otimes} \mathcal{B}(\ell_2))$ is an isometry. \square

Acknowledgements This work was completed when the first named author was visiting UNSW. He would like to express his gratitude to the School of Mathematics and Statistics of UNSW for its warm hospitality.

REFERENCES

- [1] Albiac F., Kalton N. *Topics in Banach space theory*. Graduate Texts in Mathematics, 233. Springer, New York, (2006).
- [2] Astashkin S., Sukochev F. *Orlicz sequence spaces spanned by identically distributed independent random variables in L_p -spaces*. J. Math. Anal. Appl. **413** (2014), no. 1, 1–19.
- [3] Astashkin S., Sukochev F. *Series of independent random variables in rearrangement invariant spaces: an operator approach*. Israel J. Math. **145** (2005), 125–156.
- [4] Astashkin S., Semenov E., Sukochev F. *The Banach-Saks p -property*. Math. Ann. **332** (2005), no. 4, 879–900.
- [5] Astashkin S., Sukochev F., Wong C. *Distributionally concave symmetric spaces and uniqueness of symmetric structure*. Adv. Math. **232** (2013), 399–431.

- [6] Astashkin S., Sukochev F., Wong C. *Disjointification of martingale differences and conditionally independent random variables with some applications*. Studia Math. **205** (2011), no. 2, 171–200.
- [7] Astashkin S., Sukochev F., Zanin D. *Disjointification inequalities in symmetric quasi-Banach spaces and their applications*. Pacific J. Math. **270** (2014), no. 2, 257–285.
- [8] Bekjan T. Φ -inequalities of noncommutative martingales. Rocky Mountain J. Math. **36** (2006), no. 2, 401–412.
- [9] Bekjan T., Chen Z. *Interpolation and Φ -moment inequalities of noncommutative martingales*. Probab. Theory Related Fields **152** (2012), no. 1-2, 179–206.
- [10] Bekjan T., Chen Z., Osekowski A. *Noncommutative maximal inequalities associated with convex functions*. to appear Trans. Amer. Math. Soc.(2015).
- [11] Bergh J., Löfström J. *Interpolation spaces. An introduction*. Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.
- [12] Burkholder D., Davis B., Gundy R. *Integral inequalities for convex functions of operators on martingales*. Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, pp. 223–240. Univ. California Press, Berkeley, Calif., 1972.
- [13] Burkholder D., Gundy R. *Extrapolation and interpolation of quasi-linear operators on martingales*. Acta Math. **124** (1970), 249–304.
- [14] Carothers N., Dilworth S. *Inequalities for sums of independent random variables*. Proc. Amer. Math. Soc. **104** (1988), no. 1, 221–226.
- [15] Dirksen S. *Noncommutative and vector-valued Rosenthal inequalities*. Thesis, Delft University of Technology (2011).
- [16] Dirksen S. *Noncommutative Boyd interpolation theorems*. Trans. Amer. Math. Soc.(2015).
- [17] Dirksen S., de Pagter B., Potapov D., Sukochev F. *Rosenthal inequalities in noncommutative symmetric spaces*. J. Funct. Anal. **261** (2011), no. 10, 2890–2925.
- [18] Dirksen S., Ricard E. *Some remarks on noncommutative Khintchine inequalities*. Bull. Lond. Math. Soc. **45** (2013), no. 3, 618–624.
- [19] Dodds P., Dodds T., de Pagter B. *Fully symmetric operator spaces*. Integral Equations Operator Theory **15** (1992), no. 6, 942–972.
- [20] Dykema K., Sukochev F., Zanin D. *A decomposition theorem in II_1 -factors*. J. Reine Angew. Math. (to appear).
- [21] Fack T., Kosaki H. *Generalized s -numbers of τ -measurable operators*. Pacific J. Math. **123** (1986), no. 2, 269–300.
- [22] Garsia A. *On a convex function inequality for martingales*. Ann. Probability **1** (1973), no. 1, 171–174.
- [23] Jiao Y. *Burkholder inequalities in noncommutative Lorentz spaces*. Proc. Amer. Math. Soc. **138** (2010), no. 7, 2431–2441.
- [24] Jiao Y. *Martingale inequalities in noncommutative symmetric spaces*. Arch. Math. **98** (2012), no. 1, 87–97.
- [25] Jiao Y., Sukochev F., Xie G., Zanin D. Φ -moment inequalities for independent and free independent random variables. J. Funct. Anal. **270** (2016), pp. 4558–4596.
- [26] Jiao Y., Sukochev F., Zanin D. *Johnson-Schechtman and Khintchine inequalities in noncommutative probability theory*. J. Lond. Math. Soc. (2) **94** (2016), no. 1, 113–140.
- [27] Johnson W., Schechtman G. *Martingale inequalities in rearrangement invariant function spaces*. Israel J. Math. **64** (1988), no. 3, 267–275.
- [28] Johnson W., Schechtman G. *Sums of independent random variables in rearrangement invariant function spaces*. Ann. Probab. **17** (1989), no. 2, 789–808.
- [29] Junge M. *Doob's inequality for non-commutative martingales*. J. Reine Angew. Math. **549** (2002), 149–190.
- [30] Junge M., Parcet J., Xu Q. *Rosenthal type inequalities for free chaos*. Ann. Probab. **35** (2007), no. 4, 1374–1437.
- [31] Junge M., Sukochev F., Zanin D. *Embedding of operator ideals into L_p -spaces on finite von Neumann algebra*. submitted manuscript. (2015).
- [32] Junge M., Xu, Q. *Noncommutative Burkholder/Rosenthal inequalities*. Ann. Probab. **31** (2003), no. 2, 948–995.
- [33] Kalton N., Montgomery-Smith S. *Interpolation of Banach spaces*. Handbook of the geometry of Banach spaces, Vol. 2, 1131–1175, North-Holland, Amsterdam, (2003).

- [34] Kalton N., Sukochev F., *Symmetric norms and spaces of operators*. J. Reine Angew. Math. **621** (2008), 81–121.
- [35] Krein S., Petunin Y., Semenov E. *Interpolation of linear operators*, Translations of Mathematical Monographs, Amer. Math. Soc. **54** (1982).
- [36] Lance E. *Hilbert C^* -modules. A toolkit for operator algebraists*. London Mathematical Society Lecture Note Series. **210**. Cambridge: Univ. Press, ix, 130 p.
- [37] Lindenstrauss J., Tzafriri L. *Classical Banach spaces. II. Function spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete, **97**. Springer-Verlag, Berlin-New York, 1979.
- [38] Lord S., Sukochev F., Zanin D. *Singular traces. Theory and applications*. De Gruyter Studies in Mathematics, **46**. De Gruyter, Berlin, 2013.
- [39] Lorentz G., Shimogaki T. *Interpolation theorems for the pairs of spaces (L_p, L_∞) and (L_1, L_q)* . Trans. Amer. Math. Soc. **159** (1971), 207–221.
- [40] Lust-Piquard F., Pisier G. *Noncommutative Khintchine and Paley inequalities*. Arkiv for Mat. **29** (1991): 241–260.
- [41] Maligranda L. *Indices and interpolation*. Dissert. Math., **234**. Polska Akademia Nauk, Inst. Mat., 1985.
- [42] Maligranda L. *The K -functional for p -convexifications*. Positivity. **17** (2013): 707–710.
- [43] Le Merdy C., Sukochev F. *Rademacher averages on noncommutative symmetric spaces*. J. Funct. Anal. **255** (2008), no. 12, 3329–3355.
- [44] Pisier G., Xu Q. *Non-commutative martingale inequalities*. Comm. Math. Phys. **189** (1997), no. 3, 667–698.
- [45] Rao M., Ren Z. *Applications of Orlicz spaces*. Monographs and Textbooks in Pure and Applied Mathematics, **250**. Marcel Dekker, Inc., New York, 2002.
- [46] Randrianantoanina N. *Non-commutative martingale transforms*. J. Funct. Anal. **194** (2002), no. 1, 181–212.
- [47] Randrianantoanina N. *Conditioned square functions for noncommutative martingales*. Ann. Probab. **35** (2007), no. 3, 1039–1070.
- [48] Randrianantoanina N., Wu L. *Martingale inequalities in noncommutative symmetric spaces*. J. Funct. Anal. (2015), <http://dx.doi.org/10.1016/j.jfa.2015.05.017>.
- [49] Rosenthal H. *On the subspaces of L_p ($p > 2$) spanned by sequences of independent random variables*. Israel J. Math. **8** (1970), 273–303.
- [50] Semenov E., Sukochev F. *The Banach-Saks index*. Sb. Math. **195** (2004), no. 1-2, 263–285.
- [51] Sukochev F. *Completeness of quasi-normed symmetric operator spaces*. Indag. Math. (N.S.) **25** (2014), no. 2, 376–388.
- [52] Sukochev F., Zanin D. *Johnson-Schechtman inequalities in the free probability theory*. J. Funct. Anal. **263** (2012), no. 10, 2921–2948.
- [53] Voiculescu D. *A strengthened asymptotic freeness result for random matrices with applications to free entropy*. Internat. Math. Res. Notices 1998, no. **1**, 41–63.

SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL SOUTH UNIVERSITY, CHANGSHA 410075, CHINA

E-mail address: jiaoyong@csu.edu.cn

Current address: School of Mathematics and Statistics, University of NSW, Sydney, 2052, Australia

E-mail address: yong.jiao@unsw.edu.au

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NSW, SYDNEY, 2052, AUSTRALIA

E-mail address: f.sukochev@unsw.edu.au

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NSW, SYDNEY, 2052, AUSTRALIA

E-mail address: d.zanin@unsw.edu.au

SCHOOL OF MATHEMATICS AND STATISTICS, CENTRAL SOUTH UNIVERSITY, CHANGSHA 410075, CHINA

E-mail address: zhoudejian@csu.edu.cn